Chapter 7

ROTATIONAL MOTION

GOALS
When you have mastered the contents of this chapter, you will be able to achieve the following goals:

Definitions
Define each of the following terms, and use it in an operational definition:

- angular displacement
- angular velocity
- angular acceleration
- uniformly accelerated angular motion
- torque
- center of mass
- moment of inertia
- rotational kinetic energy
- angular momentum

Equilibrium
State the conditions for static equilibrium. Rotational Motion
Write the equations for rotational motion with constant angular acceleration. Rotational Kinematics
Solve problems for systems with a fixed axis of rotation using the principles of rotational kinematics. Rotational Dynamics
Solve problems using the principles of rotational dynamics, for systems with fixed axes of rotation, including conservation of energy and conservation of angular momentum. Equilibrium Problems
Solve problems involving conditions of static equilibrium.

PREREQUISITES
Before beginning this chapter you should be familiar with Chapter 4, Forces and Newton's Laws, and Chapter 5, Energy. The quantitative aspects of rotational motion are very similar to those of kinematics (Chapter 3).
Chapter 7

ROTATIONAL MOTION

7.1 Introduction

Ever since you got up this morning you have been interacting with rotating objects, the clock hands were turning, the door knob was twisted, the water faucet was turned, the door was opened, the automobile or bicycle wheels rolled, and even if you stopped, the earth spun you through space at a high rate of speed. In fact, many people feel that going around in circles is a more common phenomena than going straight.

Since you have had considerable experience with making objects go around in circles, you already know much about rotational motion. How do you characterize the motion of a rotating object? How do you specify how fast it is rotating? Kilometers per hour hardly seems like an appropriate unit since the object really is not going any place. How do you get an object to start rotating? What are the essential features of rotation?

7.2 Angular Motion: Rotational Kinematics

The hands on the face of your watch are excellent examples of angular motion. It is upon the basis of angular displacement of the watch hands that you tell time. In one minute the second hand goes through one complete revolution, an angular displacement of $2\pi$ radians (1 revolution or 360°). See Figure 7.1. For the same period of time the minute hand has an angular displacement of one sixtieth of a revolution, or $\pi/30$ radians. Similarly, the hour hand has an angular displacement of $\pi/360$ radians in one minute. The angle between two positions of the watch hand defines an angular displacement. Angular displacement is assigned a direction parallel to the axis of rotation. The direction of the angular displacement can be found by using the right hand rule. Point the fingers of your right hand in the direction of increasing angular displacement $\Theta$. Then the thumb of your right hand points in the direction of the angular displacement vector (see Figure 7.2). The fundamental unit for angular displacement is the radian. In Figure 7.1, the direction for the angular displacement of the minute or hour hand is into the page.
The average angular velocity is defined as the time rate of change of angular displacement. Thus,

$$\omega_{\text{ave}} = \frac{\Theta_2 - \Theta_1}{t_2 - t_1} = \frac{\Delta\Theta}{\Delta t}$$

with direction given by a right-hand rule \((7.1)\)

where \(\Theta_1\) is the displacement at time \(t_1\), \(\Theta_2\) is the displacement at a later time \(t_2\), \(\Delta\Theta\) represents the change in the vector angular displacement in radians \(\Theta_2 - \Theta_1\) in time \(\Delta t\) seconds, \(t_2 - t_1\). Angular velocity is a vector quantity and has the units of radians per second \((\text{rad/sec})\). The simplest angular motion occurs if the angular velocity \(\omega\) is constant, that is

$$\Delta \Theta = \omega \Delta t$$ \hspace{1cm} (7.2)

Many of our appliances have electric motors that start from rest and soon reach a constant angular velocity for continuous operation. During the start-up period there is a change of angular velocity which is called angular acceleration. Angular acceleration is defined as the time rate of change of angular velocity. Thus,

$$\alpha_{\text{ave}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

with direction given by a right-hand rule \((7.3)\)

in which \(\Delta\omega\) represents the change in angular velocity in radians per second in time \(\Delta t\) in seconds. Angular acceleration is a vector quantity and has units of \(\text{rad/sec}^2\). Instantaneous angular velocity and acceleration are treated in the enrichment section, Section 7.11.

Let us develop the relationship for angular motion in which angular acceleration \(\alpha\) is constant. From our definition, the angular acceleration is the change in angular velocity divided by the time in which the change occurs. The magnitude of the angular acceleration is given by the following equation: \(\alpha = (\omega_f - \omega_i)/t\) where \(\omega_i\) is the magnitude of the angular velocity at time \(t\), and \(\omega_f\) is the magnitude of the initial angular velocity. We can solve this equation for the angular velocity at time \(t\),
\( \omega_f = \omega_o + \alpha t \) \hspace{1cm} (7.4)

The magnitude of the average angular velocity during time \( t \) is the sum of the final and initial values of angular velocity divided by two.

\( \omega_{\text{ave}} = (\omega_f + \omega_o) / 2 \) \hspace{1cm} (7.5)

The magnitude of the change in angular displacement in time \( t \) is found by combining Equation 7.1 with Equation 7.4:

\[ \Delta \Theta = \Theta_f - \Theta_o = \omega_{\text{ave}} t = \left[ (\omega_f + \omega_o) / 2 \right] t = \omega_o t + 1/2 \alpha t^2 \] \hspace{1cm} (7.6)

where the magnitude of the displacement at the times \( t \) and zero are given by \( \Theta_f \) and \( \Theta_o \), respectively. Another relationship, which is often used, may be obtained by eliminating \( t \) in the combination of Equations 7.4 and 7.6:

\[ 2\alpha(\Theta_f - \Theta_o) = \omega_f^2 - \omega_o^2 \] \hspace{1cm} (7.7)

If an object starts at the zero position and at rest, \( \Theta_o = 0 \) and \( \omega_o = 0 \), these equations reduce to the following:

\[ \omega_f = \alpha t \] \hspace{1cm} \[ \Theta = 1/2 \alpha t^2 \] \hspace{1cm} \[ 2\alpha \Theta = \omega_f^2 \] \hspace{1cm} (7.8)

In the chapter on kinematics we discussed the linear velocity and acceleration of a particle moving in a circle. For a rigid body rotating about a fixed axis, every point in the body is moving in a plane perpendicular to the axis of rotation and in a circle whose center is the axis. Let us now develop some relationships between the angular velocity and the angular acceleration of a rotating body and the linear velocities and linear accelerations of points within it. We shall consider only the special case of rotation about a fixed axis. Suppose we have a rigid body rotating about a fixed axis \( O \). See Figure 7.3. The point \( P \) represents any point in the body at a distance \( r \) from the axis of rotation. The path of \( P \) is a circle in the plane of \( O \) and \( P \), which is perpendicular to the axis through \( O \). The distance through which \( P \) travels, \( s \), is given by the product of the radial distance \( r \) and the magnitude of the angle of displacement \( \Theta \) in radians,

\[ s = r \Theta \] \hspace{1cm} (7.9)

As \( r \) is constant, an increment \( \Delta s \) is related to an increment of \( \Theta \) as follows:

\[ \Delta s = r \Delta \Theta \] \hspace{1cm} (7.10)

where \( \Delta \Theta \) is in radians.

If the body rotates through an angle \( \Delta \Theta \) in time \( \Delta t \), the point \( P \) moves a distance of \( \Delta s \) in the same time. We can divide both sides of Equation 7.10 by \( \Delta t \) to obtain

\[ \Delta s / \Delta t = r \Delta \Theta / \Delta t \] \hspace{1cm} (7.11)

But by definition, \( \Delta s / \Delta t = v_{\text{ave}} \) and \( \Delta \Theta / \Delta t = \omega_{\text{ave}} \) so Equation 7.10 can be rewritten as
\[ v_{ave} = r \omega_{ave} \]  

By calculus we can show a similar relationship holds for instantaneous linear velocity and instantaneous angular velocity,

\[ v_{inst} = r \omega_{inst} \]

Can you show that an object rolling without slipping has a speed of \( r\omega \)? If the object is not slipping, then the point of contact and the surface on which it is rolling are at rest with respect to one another. Then the center of the object is a distance \( r \) from the point of contact and rotating with an angular velocity of \( \omega \) with respect to the contact point. Hence, the velocity of the center is given by \( r\omega \). If \( \omega \) is constant, the angular acceleration is zero. Earlier it was shown that a particle moving in a circle at a uniform rate experiences an acceleration toward the center of the circle, called radial acceleration. In the chapter on kinematics we found the magnitude of the radial acceleration to be given by Equation 3.21,

\[ a_r = \frac{v^2}{r} \]

If we substitute \( r\omega \) for \( v \) in this expression, we get an expression for the radial acceleration in terms of the angular velocity.

\[ a_r = \omega^2 r \]

If the angular velocity is not constant, the point \( P \) is not rotating at a constant rate but has an acceleration in the direction of motion, tangential to the circle, as well as in the radial direction. This acceleration is called the tangential acceleration. To find an expression for the tangential acceleration, we begin with Equation 7.13 and change both \( v \) and \( \omega \) in a time \( \Delta t \), so \( \Delta \omega \) is the change in angular velocity and \( \Delta v \) is the corresponding change in velocity in the time interval \( \Delta t \),

\[ \frac{\Delta v}{\Delta t} = r\frac{\Delta \omega}{\Delta t} \]

where \( \frac{\Delta v}{\Delta t} \) is the average tangential acceleration \( a_t \) and \( \frac{\Delta \omega}{\Delta t} \) is the average angular acceleration \( \alpha \). The scalar form of Equation 7.15 is,

\[ a_t = r\alpha \]

an equation that applies to the motion of an object that rolls without slipping. The resultant acceleration \( a \) of \( P \) is then the vector sum of \( a_r \) and \( a_t \) (see Figure 7.4).

\[ a = a_r + a_t \]

EXAMPLES

1. Suppose you are riding on a merry-go-round which is making 3.00 revolutions per minute and you are moving in a circle with a radius of 6.00 m. What is your linear speed?

\[ \omega (\text{rad/sec}) = \frac{\Delta \theta}{\Delta t} = 3.00 \times 2\pi/60 = \pi/10.0 \text{ rad/sec} \]
\( v = \omega r = (\pi/10)(6.00) = 6.00\pi/10.0 = 1.88 \text{ m/sec} \) \hspace{1cm} (7.12)

If the merry-go-round is speeded up to 6.00 rpm in 1.00 minute what is the angular acceleration? What is the tangential acceleration during this minute? What would be the total acceleration when the merry-go-round was going 5.00 rpm?

The angular acceleration can be calculated by using the definition of \( \alpha \):

\[ \alpha = \frac{\Delta \omega}{\Delta t} = \frac{(12.0\pi \text{ / min} - 6.00\pi \text{ / min})}{60 \text{ sec} \times 1 \text{ min}/60 \text{ sec}} \]

\[ = \frac{(12.0\pi - 6.00\pi)}{(60 \times 60)} = \frac{6.00\pi}{3600} = \frac{\pi}{600} = 5.24 \times 10^{-3} \text{ rad/sec}^2 \] \hspace{1cm} (7.3)

The tangential acceleration can be obtained from Equation 7.16:

\[ a_t = r\alpha = 6.00 \times \pi / 600 = \frac{\pi}{100} = 3.14 \times 10^{-2} \text{ m/sec} \] \hspace{1cm} (7.16)

\[ \omega = (5.00 \times 2\pi) / 60.0 = \pi / 6.00 = 5.24 \times 10^{-1} \text{ rad/sec} \]

\[ a_r = \omega^2 r = \left[ (\pi^2 \text{ rad}^2 / (6.00)^2 \text{ sec}^2) \right] (10.0 \text{ m}) = 10\pi^2 / 360 = 2.74 \text{ m/sec}^2 \] \hspace{1cm} (7.14)

The magnitude of the total acceleration is equal to vector sum of \( a_r \) and \( a_t \):

\[ a = \sqrt{a_r^2 + a_t^2} = 2.74 \text{ m/sec}^2 \]

2. One reads on an electric motor name plate that its operating speed is 1800 rpm. Assume you find that it takes 10.0 sec to reach its operating speed. Find the final angular displacement during start-up. Assume constant angular acceleration.

\[ 1800 \text{ rpm} = 30.0 \text{ rps} \hspace{1cm} \omega_i = 30.0 \text{ rps} = 60.0\pi \text{ rad/sec} \]

\[ \alpha = [\omega_f - \omega_o] / t = [(60.0\pi - 0)\text{rad/sec}] / 10 \text{ sec} = 6.00\pi \text{ rad/sec}^2 \]

\[ \Theta = (1/2)\alpha t^2 = 1/2 \times 6.00\pi \text{ rad/sec}^2 \times 100 \text{ sec}^2 = 300\pi \text{ rad} \]

\[ \Theta = 300\pi / 2\pi = 150 \text{ revolutions} \]

If you recall your study of kinematics you will notice the great similarity between the concepts we have developed to explain rotational motion and the ones used in Chapter 3 for linear motion. If you mentally bend linear motion into a circular track, it would be the same as rotational motion. In the table below we display the various equations side-by-side to emphasize their similarities.

<table>
<thead>
<tr>
<th>Concept of Linear and Rotational Motion</th>
<th>Linear Dynamics</th>
<th>Rotational Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical Quantity/Rectilinear Motion</td>
<td>Physical Quantity/Rotation about a Fixed Axis</td>
<td></td>
</tr>
<tr>
<td>Mass/( m )</td>
<td>Moment of inertia/( I )</td>
<td></td>
</tr>
<tr>
<td>Force/( F = ma )</td>
<td>Torque/( \tau = I\alpha )</td>
<td></td>
</tr>
<tr>
<td>Work/( W = Fd )</td>
<td>Work/( W = \tau \theta )</td>
<td></td>
</tr>
<tr>
<td>Translational KE/( KE = \frac{1}{2}mv^2 )</td>
<td>Rotational KE/( KE = \frac{1}{2}I\omega^2 )</td>
<td></td>
</tr>
<tr>
<td>Linear momentum/( p = mv )</td>
<td>Angular momentum/( L = I\omega )</td>
<td></td>
</tr>
</tbody>
</table>
7.3 Torque

Let us look at the physics of opening a hinged door (see Figure 7.5). As part of life's experiences you have learned to carry out this operation, but can you explain the physics of the operation? You observe that there are hinges on one edge of the door, and there is a latch on the other edge. The type of motion that occurs as a door is opened is rotation about a fixed axis. Assume that the latch is released. You know that you must exert a force on the door to open it. Where must the force be applied, and what is its direction? By doing experiments you would find that you cannot open the door by applying the force through the hinges regardless of the size of the force. This seems to indicate that there is a most desirable position to apply the force. You probably say to apply the force in a direction perpendicular to the door, but not through the axis of the hinge. Again, upon the basis of experience, you would probably apply the force near the edge opposite the hinges. Why? In order to produce a rotational motion you must apply a torque. The magnitude of the torque or moment of force applied to open the door is given by the product of the force and its lever arm.

The lever arm is equal to the perpendicular distance from the axis of rotation (the hinge of the door) to the direction of the force. The magnitude of the torque of force F about the axis of the hinge is given by

\[ \tau = Fd \quad (7.18) \]

where \( \tau \) is the magnitude of the torque, \( F \) is the magnitude of the force applied and \( d \) is the perpendicular distance from the force to the hinge. The dimensions of torque are \( ML^2 T^{-2} \), and the units are newton-meters in the SI system, or dyne-centimeters in the cgs system.

The torque of a force is a vector quantity as it has both magnitude and direction. The direction of rotation is either clockwise or counterclockwise. Torque is represented by a vector in which the line of action is through the axis of rotation, in the case of the door it is the hinges. The length of the vector represents the magnitude. Its direction is
indicated by clockwise motion as one looks along the axis of rotation. Once again the right-hand algorithm can be used for finding the direction of the torque vector. Point the fingers of your right hand in the direction of the force with your palm open toward the axis of rotation. Then your right thumb points in the direction of the torque (see Figure 7.6). The torque in the figure would be represented by a vector pointing upward with a length proportional to \( Fd \sin \theta \).

**FIGURE 7.6**
A torque is a vector product of two vectors. To determine the direction of the torque vector one uses the right hand. If the fingers of the right hand go around the axis in the direction of rotation, the thumb will then point along the axis in the direction of torque vector.

7.4 Center of Mass and Equilibrium

In your younger days you probably improvised a teeter-totter board. You tried to "balance" it on the axis of support. In its balanced position the torque tending to produce counterclockwise motion is equal to the torque tending to produce clockwise motion. A vertical line (direction of plumb-bob line) through the axis of support will pass through the center of mass of the body. The center of mass of a rigid body is its balance point. For the teeter-totter board the support acts upward with a force of \( R \) which is equal to the weight of the board. See Figure 7.7. We can say the force of gravity (weight) produces zero torque about the center of mass. Thus for a body, you can consider all of its weight as acting at the center of mass. For a uniform body, this coincides with the geometric center. You can determine the center of mass of a body experimentally by finding its point of balance. You can also determine the center of mass of a body such as a plate by hanging the body at different positions and establishing plumb-bob lines for each point of suspension. The point of intersection of the plumb-bob lines determines the center of mass of the body (see Figure 7.8).
In a formal way the center of mass of a system consisting of n particle masses is determined by equating the sum of the mass moments of the particles of a system to the moment of the total mass located at the center of mass for a given axis of rotation. The moment of a point mass is defined as the product of the mass times the distance from the point to the axis of rotation,

\[ m_1 r_1 + m_2 r_2 + m_3 r_3 + \ldots + m_n r_n = \sum_{i=1}^{n} m_i r_i = Mr_{ave} \]  \hspace{1cm} (7.19)

where the symbol \( \sum_{i=1}^{n} \) means to add the terms from the first to the n\textsuperscript{th}. The term \( m_i \) is the mass of the i\textsuperscript{th} particle, \( r_i \) is the distance of the i\textsuperscript{th} particle from the reference axis, \( r_{ave} \) is the location of the center of mass of the system from the reference axis, and \( M \) is the total mass of all the n particles in the system:

\[ M = \sum_{i=1}^{n} m_i \]

For example, if you are given four equal masses m located as shown in Figure 7.9,

\[ m \times 1.00 \text{ cm} + m \times 4.00 \text{ cm} + m \times (-2.00 \text{ cm}) + m \times (-7.00 \text{ cm}) = -4 \text{ m} r_{ave} \]

\[ r_{ave} = -1.00 \text{ cm} \]

that is, the center of the mass is 1 cm to the left of the reference axis. It is important to note that the choice of axis location does not change the center of mass of a system. Consider the axis through the mass at the far left. One then has

\[ 0.00 \times m + m \times 5.00 \text{ cm} + m \times 8.00 \text{ cm} + m \times 11.0 \text{ cm} = 4.00 \text{ m} r_{ave} \]

\[ 24.0 \text{ m cm} = 4 \text{ m} r_{ave} \]

\[ r_{ave} = 6.00 \text{ cm} \]

Hence, the center of mass is 6 cm to the right of the far-left mass. This is exactly the same position as indicated in the first calculation.

**EXAMPLE**

Suppose you have a uniform 4.00-m beam weighing 500 N, and it has a load of 200 N suspended 1.00 m from the left end, one of 300 N at 3.00 m from the left end, and that the system is supported by one hanger (see Figure 7.10).

Where should the hanger be placed? It must be at the center of mass of the system.
Take the left end of the beam as a reference point, then, for moments about $O$,
\[
r_{\text{ave}} = \frac{(200 \text{ N})(1.00 \text{ m}) + (500 \text{ N})(2.00 \text{ m}) + (300 \text{ N})(3.00 \text{ m})}{(200 \text{ N} + 500 \text{ N} + 300 \text{ N})} = 2.1 \text{ m}
\]
The center of mass of the system is at 2.1 m from the left end.

We are now ready to consider the conditions necessary for equilibrium.

A system is said to be in a state of equilibrium if:

1. The sum of the forces acting on the system in any direction is zero (no translational acceleration) (see Chapter 4).
2. The sum of the torques acting on the system (moment of forces) about any axis is zero (no angular acceleration).

EXAMPLES

1. A uniform 4.00-m plank which weighs 200 N is to be used for a teeter-totter board for two girls - one weighing 300 N and one weighing 240 N. Each is to be seated 0.250 m from the end (see Figure 7.11). For equilibrium, where should the fulcrum be placed and how much force must it exert?

Make a free body diagram. For a condition of equilibrium, $R$, the upward force, must be equal to the total of the downward forces $= 240 + 200 + 300 = 740 \text{ N}$. The fulcrum must support 740 N. Consider $A$ as the point about which moments are to be considered. For the second condition of equilibrium to be satisfied, the clockwise torques must be equal to the counterclockwise torques. Then,
\[
(240 \text{ N})(0.250 \text{ m}) + (200 \text{ N})(2.00 \text{ m}) + (300 \text{ N})(3.75 \text{ m}) = 740 \, r_{\text{ave}}
\]
\[
r_{\text{ave}} = \frac{1585}{740} = 2.15 \text{ m}
\]
The fulcrum should be placed at 2.15 m from the end of the plank near the smaller girl.
2. A crane is used to support a load. The uniform boom $AB$ weighs $2.00 \times 10^3$ N, and when the boom makes an angle of $37^\circ$ with the vertical, the tie $CB$ is horizontal (see Figure 7.12). Find the tension in the cable, the compression in the boom $AB$, and the reaction force $F$ at $A$. Is the reaction force at $A$ along the boom?

We begin by making a free body diagram of the boom as in Figure 7.12b. The force equations are:

- $F_v = 2.00 \times 10^3$ N + $8.00 \times 10^3$ N = $1.00 \times 10^4$ N for vertical components
- $F_h = T$ for horizontal components

Set up a moment equation about A:

$$T = \frac{(2.0 \times 10^3 \text{ N})(1.50 \text{ m}) + (8.00 \times 10^3 \text{ N})(3.00 \text{ m})}{4.00 \text{ m}} = (4.00 \text{ m}) T$$

Then

$$F_h = 6.75 \times 10^3 \text{ N}$$

$$F = \sqrt{R[T(2.00 \times 10^3 \text{ N})^2 + (6.75 \times 10^3 \text{ N})^2]}$$

$$F = 1.21 \times 10^4 \text{ N}$$

$$\tan \Theta_F = \frac{F_v}{F_h} = 1.00 \times 10^4 / 6.75 \times 10^3 = 1.48$$

$$\Theta_F = 56^\circ$$

$F$ is not along the boom but is at a greater angle with the horizontal. To get the compression $C$ in the boom, let us consider the forces acting at $B$ (See Figure 7.12c).

$$C \sin \Theta = 8.00 \times 10^3 \text{ N} \quad c \cos \Theta = 6.75 \times 10^3 \text{ N}$$

$$C = \sqrt{R[(8.00 \times 10^3 \text{ N})^2 + (6.75 \times 10^3 \text{ N})^2]}$$

$$C = 1.05 \times 10^4 \text{ N}$$
3. The human leg provides another example of an equilibrium problem. Let’s assume you are standing on your right leg. See Figure 7.13a. Consider the acetabulum and upper end of the femur as a frictionless ball and socket joint. $F$ is the force acting on the greater trochanter by the hip abductor muscle at an angle $\Theta$ with the horizontal, and $R$ is the force the acetabulum exerts on the head of the femur. Given $\Theta = 70^\circ$. Find the magnitude of $F$ and $R$.

The leg is in equilibrium under the following forces: the normal forces of floor on your foot $N$, the weight of the leg $W_L$, the force $F$, and the force $R$. Let us draw a free body diagram of the leg, Figure 7.13b.

In order for you to stand on one leg, the force of the floor on your foot must be upward through your center of gravity and equal to your weight. We can write three equations for the equilibrium conditions.

We can resolve $R$ into its horizontal and vertical components.

$$R_h - F \cos 70^\circ = 0$$ horizontal components

$$N - W_L - R_v + F \sin 70^\circ = 0$$ vertical components

As the joint is frictionless and round, the force $R$ acts through the center of spherical surface and makes an angle $\phi$ with the horizontal. Set up the moment equation about this point,

$$N \times 11.0 \text{ cm} = W_L \times 3.00 \text{ cm} + (F \sin 70^\circ) \times 7.00 \text{ cm}$$

These are three equations that can be solved if there are only three unknowns. In order to complete a numerical solution for $F$ and $R$ you need to know your weight and the weight of your leg. Let’s assume your weight is 600 N (mass of about 60.0 kg) and that your right leg weight is 15 percent of your total weight, or about 90.0 N. Substituting these values in moment equation we can solve the problem:

$$600 \text{ N} \times 11.0 \text{ cm} = 90.0 \text{ N} \times 3.00 \text{ cm} + 7.00 \text{ cm} \times 0.940 F$$

$$F = 962 \text{ N}$$ the force exerted by the abductor muscle
\( R_h = 962 \times 0.326 = 313 \text{ N} \)
\( R_v = 680 - 90 + 962 \times 0.940 = 1.43 \times 10^3 \text{ N} \)
\( \tan \phi = \frac{R_v}{R_h} = \frac{R_v}{313} = 4.58 \)
\( \phi = 77.6^\circ \)
\( R = \sqrt{R_h^2 + R_v^2} = \sqrt{(313)^2 + (1.43 \times 10^3)^2} \)
\( = 1.47 \times 10^3 \text{ N} = \text{force exerted by the acetabulum.} \)

### 7.5 Rotational Dynamics

You may recall that in Chapter 4 you found that if an unbalanced force is acting upon a body, an acceleration is produced (Newton’s second law of motion). If there is an unbalanced torque acting upon a body, there is an angular acceleration produced. In fact, one can state a law which is parallel to Newton’s second law of motion:

*If an unbalanced torque is acting upon a body, an angular acceleration is produced that is proportional to the torque and in the direction of the torque.*

If you apply an equivalent torque to a different body, you may find that a different value of angular acceleration is produced. The value for \( \alpha \) for a given torque depends upon the distribution of the mass of the body relative to the axis of rotation. You can write an equation setting the torque equal to the product of a constant of proportionality times the angular acceleration,

\[ \tau = I \alpha \]  \hspace{1cm} (7.20)

where \( I \) is the moment of inertia of a body and its value depends upon the distribution of its mass relative to the axis of rotation. If we have a system made up of point masses \( m_1, m_2, \ldots, m_n \) as shown in Figure 7.14 the moment of inertia of this system about \( O \) is defined by the following equation:

\[ I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2 \]  \hspace{1cm} (7.21)

where \( r_1, r_2, r_3, \ldots, r_n \) are the distances of the various point masses from the axis of rotation, \( O \). If we consider the moment of inertia of a system about a different axis of rotation, we will generally find it to have different moment of inertia. In general, we can write

\[ I = \sum_{i=1}^{n} m_i r_i^2 \]  \hspace{1cm} (7.22)

where the symbol sigma \( \sum_{i=1}^{n} \) means to add the terms from the first to the \( n^{th} \), where each term is given by the product of a point mass times the square of its distance from the axis of rotation.

For a hoop rotating about its geometric center (Figure 7.15), its moment of inertia is \( m r^2 \) where \( m \) is its mass, and \( r \) is its radius:

\[ I = m r^2 \]

The moment of inertia has the dimensions of \( \text{ML}^2 \) and units of \( \text{kg} \cdot \text{m}^2 \) in the SI system. For solid bodies one uses the methods of calculus to determine the moment of inertia of a body. About the geometric axis the moment of inertia of a solid cylinder is \( (1/2)mr^2 \)
and for a solid sphere it is \((2/5)mr^2\). However, the moment of inertia has a different value about any other axis. The moment of inertia about the axis parallel to the axis of rotation through the center of mass is given by

\[ I = I_G + md^2 \]  
(7.23)

(known as the parallel axis theorem) where \(I_G\) is the moment through the center of mass, \(I\) is the moment of inertia about a parallel axis and at a distance of \(d\) from the center of mass axis.

**EXAMPLES**

1. Suppose that we have a solid disk of mass 5.00 kg and a radius 0.200 m mounted on frictionless bearings. A constant force of 5.00 N is applied by a light ribbon wrapped around the rim of the disk. What is the angular acceleration of the disk?

\[
I = \left(\frac{1}{2}\right) mr^2 = \left(\frac{1}{2}\right)(5.00\, \text{kg})(0.200\, \text{m})^2 = 0.100\, \text{kg}\cdot\text{m}^2
\]

\[
\tau = TR = I\alpha
\]

\[
\alpha = 10.0\, \text{rad/ sec}^2
\]

What mass hanging on the ribbon would produce the same angular acceleration? The tension in the ribbon is to be 5.00 N.

\[
T = mg - ma = ra = 0.200\, \text{m} \times 10.0\, \text{rad/sec} = 2\, \text{m/ sec}^2
\]

\[
T = 5.00\, \text{N} = m(9.80 - 2.00)\, \text{m/sec}^2
\]

\[
m = 0.640\, \text{kg}
\]

2. Recent developments of high-speed flywheels have led to the use of flywheels as energy storage devices. One such use involves using flywheels in each car of a train to store energy during the braking process. This energy increases the speed of the flywheel during braking and then this energy can be used in accelerating the train when it starts up again. Let us consider some typical data. We want to find the moment of inertia of the flywheel when a torque of \(17.0\pi\text{N-m}\) is applied to accelerate it uniformly from 10,200 rpm to 15,000 rpm in 80.0 sec.

\[
\alpha = \frac{\Delta \omega}{\Delta t} = \left[\frac{4800\, \text{rev/min} \times 2\pi\text{rad/rev} \times 1\, \text{min}/60\, \text{sec}}{80.0\, \text{sec}}\right] = 2.00\pi\text{rad/ sec}^2
\]

\[
\tau = I\alpha = I \times 2.00\pi\text{rad/ sec}^2 = 17.0\pi\text{N-m}
\]

\[
I = 8.50\, \text{kg}\cdot\text{m}^2
\]

### 7.6 Rotational Work

Work is defined as the product of force times the distance through which it acts. If the force acts at a perpendicular distance \(R\) from the axis rotation (Figure 7.16), the distance through which \(\mathbf{F}\) acts is \(R\theta\) where \(\theta\) is the angle of rotation in radians. So we compute the work as the product of the magnitude of the force times \(R\theta\), but force times \(R\) is equal to the magnitude of the torque \(\tau\),

\[
\text{work} = \mathbf{F} \cdot \theta
\]

or in vector form,

\[
= \tau \theta(7.24)
\]

You notice the similarity between this equation and our previous Equation 5.1 for computing the work done by a force.

\[
w = \mathbf{F} \cdot s
\]

**EXAMPLE**

How much work does a girl do in pedaling a bicycle for one minute if the pedal arm is 20.0 cm, a constant force of 10.0 N is exerted on the pedal, and the pedals make 60.0 revolutions per minute? Assume the force is active on each pedal for one-half of the time.
The magnitude of the torque acting to produce rotation is
\[ \tau = FR = 10.0 \text{ N} \times 0.200 \text{ m} = 2.00 \text{ N}-\text{m} \]
\[ \Theta = 2\pi \text{ 60.0 rpm} = 120\pi \text{ rad/min} \]
\[ w = \tau \Theta = 2.00 \text{ N}-\text{m} \times 120\pi \text{ rad/min} = 240\pi \text{joules/min} \]
\[ w = 754 \text{ J} \text{ in one minute} \]

### 7.7 Kinetic Energy of Rotation

If the torque is producing an angular acceleration, there is a change in the kinetic energy of the rotating body. We can derive an expression for the kinetic energy of rotation,

\[ \text{Work} = \tau \Theta = I\alpha \Theta = \text{the change in kinetic energy} \quad (7.25) \]

From the equations of uniform angular acceleration we know that the square of the angular velocity is equal to twice the product of the angular acceleration times the angle.

\[ \alpha \Theta = \omega_f^2 / 2 - \omega_o^2 / 2 \]

If we substitute this expression for \( \alpha \Theta \) in Equation 7.25 we obtain another expression for the change in the kinetic energy of rotation:

\[ \text{change in KE rotation} = \left( \frac{1}{2} \right) I (\omega_f^2 - \omega_o^2) \quad (7.26) \]

Again, note the parallel to the expression for the change in the kinetic energy of translation, \( KE = \left( \frac{1}{2} \right) m(v_f^2 - v_o^2) \).

A rolling solid is an example of a body that has both linear and angular motions simultaneously. Specifically, let us consider the case of the front wheel on a moving bicycle. The wheel rotates about its axle and the axle advances in a forward direction. In such cases it is possible to consider the kinetic energy in two parts:

1. that due to the translation of the center of mass, that is, \( \left( \frac{1}{2} \right) mv^2 \)
2. that due to rotation about an axis through the center of mass, that is, \( \left( \frac{1}{2} \right) I\omega^2 \)

Thus the total kinetic energy of a rolling body is given by the sum of the kinetic energy of its translation, plus the rotational kinetic energy.

\[ \text{KE total} = \text{KE translation} + \text{KE rotation} \]
\[ \text{KE total} = \left( \frac{1}{2} \right) mv^2 + \left( \frac{1}{2} \right) I\omega^2 \quad (7.27) \]
EXAMPLE

Let us consider the relative distribution of the kinetic energy between kinetic energy of translation and kinetic energy of rotation for various objects. Suppose that one has three objects, a hoop, a solid cylinder, and a sphere of equal mass. These are raised to the top of a common incline (Figure 7.17). In this position all of the objects would have the same potential energy (mg\ h) relative to the foot of the incline. If each body were permitted to roll down the incline (this means there is enough friction to produce rolling and no sliding), and if there is no loss of energy resulting from rolling friction, then the total kinetic energy at the base should be the same for each body and equal to the potential energy at the top.
The general equation is then:

\[ mgh = \left( \frac{1}{2} \right) mv^2 + \left( \frac{1}{2} \right) I \omega^2 \]  

(7.28)

This equation can be rewritten for each object putting in its value for the moment of inertia:

- for the sphere  \[ mgh = \left( \frac{1}{2} \right) mv^2 + \left( \frac{1}{2} \right) \left( \frac{2mr^2}{5} \right) \omega^2 \]
- for the cylinder  \[ mgh = \left( \frac{1}{2} \right) mv^2 + \left( \frac{1}{2} \right) \left( \frac{mr^2}{2} \right) \omega^2 \]
- for the hoop  \[ mgh = \left( \frac{1}{2} \right) mv^2 + \left( \frac{1}{2} \right) \left( \frac{mr^2}{2} \right) \omega^2 \]

As previously shown for objects rolling without slipping, \( v = \omega r \). Note in each case that the velocity at the base is independent of the mass but that it does depend upon the distribution of the mass. That is, it depends upon \( I \). Solving these equations for \( v \), one finds that velocity for each object is given by a different expression:

- velocity of sphere = \( \left( \frac{10}{7} \right) (gh)^{1/2} \)
- velocity of cylinder = \( \left( \frac{4}{3} \right) (gh)^{1/2} \)
- velocity of hoop = \( (gh)^{1/2} \)

The sphere will require the least time of the three bodies to roll down the plane and the hoop will have the largest percentage of the total kinetic energy in rotational kinetic energy. You should be able to show that for this example half of the total kinetic energy of the hoop is rotational, one-third of the kinetic energy of the solid cylinder is rotational and two-sevenths of the kinetic energy of the sphere is rotational kinetic energy.
7.8 Angular Momentum

Have you watched the performance of a skilled ice skater? You may have seen the skater begin a slow spin maneuver with arms and legs extended (Figure 7.18). Then the skater slowly brings her arms and legs together. As she does this, her rate of spin increases until she has made herself as “thin” as possible and her spin rate has reached a maximum value. Let us discuss the phenomenon in the language of this chapter. At the beginning of her maneuver, the skater with extended limbs has a relatively large amount of inertia \(I \) large) and her angular velocity is small \(\omega \) small). As she changes the position of her body and limbs, her moment of inertia becomes smaller \(I \) small), and her angular velocity becomes large \(\omega \) large). If we neglect any friction between the ice skates and the ice, then the skater is an isolated system acted upon only by the vertical force of gravity. If her initial axis of spin is also vertical then the gravitational forces do not apply any torque that will change the rate of rotation of the skater. How then does her spin rate change?

In a way analogous to our definition of linear momentum \(p = mv\), we can assume that a rotating body has angular momentum \(L\) which we will define as equal in magnitude to the product of the moment of inertia and the angular velocity,

\[
L = I \omega
\]

(7.29)

where \(L\) is the angular momentum vector whose direction will be in the direction of the angular velocity vector \(\omega\) for simple cases. The direction of \(L\) can be found using a right-hand algorithm (see Figure 7.19).

The line of the vector \(L\) is parallel to the axis of rotation, and the direction is such that one sees clockwise rotation as looking along the axis.

Angular momentum is conserved if there is no unbalanced torque acting on the system. Then the product of the moment of inertia and the angular velocity will remain a conserved quantity of the system,

\[
I_f \omega_f = I_i \omega_i
\]

(7.30)

where the subscripts \(i\) and \(f\) refer to the initial and final states of the system and \(I\) and \(\omega\) represent the moment of inertia and the angular velocity of the system. A typical example of this conservation law is the skater we discussed above. By reducing her moment of inertia, she increases her angular velocity. She comes out of the spin by increasing her moment of inertia and thus reducing her angular velocity.
EXAMPLES

1. Let us consider the case of the skater who goes into a spin. Assume that at the beginning of the spin her moment of inertia is 10.0 kg·m² and the angular velocity is 5.00 rad/sec. If the moment of inertia of the skater is reduced to 2.00 kg·m², what is her final angular velocity?

\[ I_i \omega_i = I_f \omega_f \]

\[(10.0 \text{ kg·m}^2) (5.00 \text{ rad/sec}) = (2.00 \text{ kg·m}^2) \omega_f \]

\[ \omega_f = 25.0 \text{ rad/sec} \]

2. Assume a high diver is rotating at a slow rate (10.0 rpm) in an outstretched position. The diver goes into a tight tuck position. We want to estimate the new rate of rotation for the tuck position. We will assume that in the initial position we can approximate the diver by a rod rotating about its center of mass (moment of inertia = \( (1/12) ML^2 \), where \( M = \text{mass} \) and \( L = \text{Length} \)). In the tight tuck position we assume that the diver approximates a disk with a radius of \( L/4 \). Such a disk (same mass as "rod") has a moment of inertia of \( (1/2) M (L/4)^2 = (1/32) ML^2 \). Since there are no external torques acting on the diver, the angular momentum of the diver must be conserved as expressed in Equation 7.30. Solving for the angular velocity in the tuck position we find:

\[ \omega_i = l_i \omega_i / l_i = (1/12) ML^2 \omega_i / (1/32) ML^2 = 32 \omega_i / 12 \]

\[ \omega_i = 2.67 \omega_i = 2.67 \times 10.0 \text{ rpm} = 26.7 \text{ rpm} \]

We have now concluded the definitions of various concept related to rotational motion. We summarize all these concepts and compare them with their linear motion analogs in Table 7.1.

<table>
<thead>
<tr>
<th>concepts of Linear and Rotational Motion</th>
<th>Linear Dynamics</th>
<th>Rotational Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical Quantity</td>
<td>Rectilinear Motion</td>
<td>Physical Quantity</td>
</tr>
<tr>
<td>Mass</td>
<td>( m )</td>
<td>Moment of inertia</td>
</tr>
<tr>
<td>Force</td>
<td>( F = ma )</td>
<td>Torque</td>
</tr>
<tr>
<td>Work</td>
<td>( W = Fd )</td>
<td>Work</td>
</tr>
<tr>
<td>Translational KE</td>
<td>( KE = \frac{1}{2} mv^2 )</td>
<td>Rotational KE</td>
</tr>
<tr>
<td>Linear momentum</td>
<td>( p = mv )</td>
<td>Angular momentum</td>
</tr>
</tbody>
</table>

7.9 Conservation of Energy and Momentum

The principle of conservation of energy applies to bodies with rotational motion as well as to those in translation. Also we have the conservation of angular momentum as well as the conservation of linear momentum in isolated systems. In fact, the conservation of angular momentum may be considered a more general law than the conservation of linear momentum. Linear momentum is conserved for all systems on which no external forces are acting. Angular momentum is conserved for all systems on which there are no external torques acting. It is possible to have forces acting on a system that produce no torque. It is not possible to have a torque produced with no forces acting. Hence, the class of all systems that conserve angular momentum is larger than the class of systems that conserve linear momentum.
Let us summarize the three conservation laws we have discussed in Chapter 5, 6, and 7. For systems upon which no net work is done the total energy of the system is conserved.

\[
\text{total energy} = \text{constant}
\]

For systems with both kinetic and potential energy, the sum of the kinetic energy and the potential energy remains constant,

\[
\text{KE} + \text{PE} = \text{constant}
\]

For systems near the surface of the earth,

\[
\left(\frac{1}{2}\right) m v^2 + mgh = \text{constant}
\]

where \( m \) is mass (kg), \( v \) is velocity (m/sec), \( g \) is gravitational acceleration (9.80 m/sec\(^2\)), and \( h \) is the distance (m) above a reference line.

For systems upon which no net external forces are acting, the linear momentum \( \mathbf{p} \) is conserved

\[
\text{linear momentum} = \text{constant}
\]

\[
\mathbf{p} = m \mathbf{v} = \text{constant}
\]

where \( m \) is the mass of the system and \( \mathbf{v} \) is the velocity of the center of mass of the system. You will notice that this conservation law is a vector law, and so it may be expressed as three independent scalar equations involving each of the three components of the linear momentum, \( p_x, p_y, \) and \( p_z \).

For systems upon which no net external torques are acting, the angular momentum \( \mathbf{L} \) is conserved,

\[
\text{angular momentum} = \text{constant}
\]

\[
\mathbf{L} = I \mathbf{\omega} = \text{constant}
\]

where \( I \) is the momentum of inertia of the system and \( \mathbf{\omega} \) is the angular velocity of the system. This is also a conservation law that involves vector quantities; so the three components of the angular momentum, \( L_x, L_y, \) and \( L_z \), must be constant for a system whose angular momentum remains constant. Notice that angular momentum is conserved for all systems that experience net forces that act through the center of rotation of the system since such forces would not exert any torque upon the system. One example of such a system is the model of our solar system, with the sun as the fixed origin of the rotation of the planets around the sun and which considers only the gravitational force between the sun and each planet. The line of action of the force of attraction between the sun and each planet passes through the sun, the assumed axis of rotation, so the gravitational attraction exerts no torque on the planet. Hence, angular momentum of a planet in orbit around the sun must remain constant.

These three conservation laws can be used to solve problems of particle dynamics in situations where the use of Newton's laws of motion would be quite difficult. Energy, linear momentum, and angular momentum are mental constructs that we can use to explain a wide variety of physical phenomena and to give ourselves a feeling of understanding the universe.
EXAMPLE

A hoop rolls down an inclined plane which has an elevation of 1.00 m. What is its velocity at the foot of the inclined plane? Neglect frictional losses.

At the top of the inclined plane the hoop has potential energy, and at the foot of the plane it has kinetic energy. The potential energy at the top is equal to the kinetic energy at the foot of the plane:

\[ PE_{\text{top}} = mgh \]
\[ KE_{\text{foot}} = \left(\frac{1}{2}\right)mv^2 + \left(\frac{1}{2}\right)I\omega^2 \]

It has kinetic energy both of translation and rotation.

The moment of inertia of a hoop is equal to \( mr^2 \), giving

\[ mgh = \left(\frac{1}{2}\right)mv^2 + \left(\frac{1}{2}\right)(mr^2)\omega^2 \]

From this one sees that each term is divisible by \( m \), which means the velocity is independent of its mass: \( gh = \frac{1}{2}v^2 + \frac{1}{2}r^2 \omega^2 \), where \( \omega \) is equal to \( v \) if the hoop rolls without slipping; thus one sees the velocity is independent of the radius. Hence, \( gh = \left(\frac{1}{2}\right)v^2 + \left(\frac{1}{2}\right)v^2 = v^2 \). This also tells us that the kinetic energy of rotation equals the kinetic energy of translation for the hoop and that

\[ v = \sqrt{gh} = \sqrt{9.80 \times 1.00} = \sqrt{9.80} \text{ m/sec} \]

If the hoop had slid down a frictionless incline, the velocity would have been

\[ v = \sqrt{2gh} \]

7.10 Planetary Motion

The apparent zig-zag motion of the planets across the night sky puzzled human observers of the heavens for many centuries. We now have the mental constructs to discuss planetary motion in a simple way. Consider the sun as a large, massive body that produces a gravitational field. The strength of the field is given by the law of gravitational attraction, \( F/m = GM/r^2 \) where \( F \) is the magnitude of the force of gravitational attraction, \( G \) is the universal gravitational constant, \( M \) is the mass of the sun, \( m \) is the mass of the object in the sun’s gravitational field, and \( r \) is the distance between the object and the sun. The gravitational force acts toward the sun. It applies no torque to an object moving in a path around the sun. So the angular momentum of objects moving in closed orbits around the sun is constant,

\[ L = I \omega = (mr^2)(v/r) = mvr = \text{constant} \quad (7.31) \]

Although the planets travel in elliptical orbits with the sun at one focus according to Kepler's first law of planetary motion, let us assume that the planets travel in circular orbits. The result we will derive is also true for elliptical orbits, but the mathematics is simpler for circular orbits. Then we can show that Equation 7.32 is equivalent to Kepler's second law of planetary motion which states; the line joining the sun and a planet sweeps out equal areas in equal times. For circular motion \( v \) and \( r \) are perpendicular to each other and can be drawn as the two sides of a right triangle (see Figure 7.20). The area of such a triangle is proportional to the product of velocity times radius, but \( vr \) is proportional to the angular momentum, so
area $\propto vr = L/m = \text{constant}$ \hfill (7.32)

Kepler's third law can also easily be deduced for circular orbits. This law states that the ratio of the cube of the radius of the orbit to the square of the time for the planet to make a complete cycle around the sun is a constant for all the planets (see Table 7.2).

<table>
<thead>
<tr>
<th>Planet</th>
<th>Mean Distance to the Sun $R$ (m)</th>
<th>Period of Time for One Cycle $T$ (sec)</th>
<th>$R^3/T^2$ (m$^3$/sec$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>$5.79 \times 10^{10}$</td>
<td>$7.60 \times 10^4$</td>
<td>$3.36 \times 10^{18}$</td>
</tr>
<tr>
<td>Venus</td>
<td>$1.08 \times 10^{11}$</td>
<td>$1.94 \times 10^7$</td>
<td>$3.35 \times 10^{18}$</td>
</tr>
<tr>
<td>Earth</td>
<td>$1.50 \times 10^{11}$</td>
<td>$3.16 \times 10^7$</td>
<td>$3.38 \times 10^{18}$</td>
</tr>
<tr>
<td>Mars</td>
<td>$2.28 \times 10^{11}$</td>
<td>$5.94 \times 10^7$</td>
<td>$3.36 \times 10^{18}$</td>
</tr>
<tr>
<td>Jupiter</td>
<td>$7.78 \times 10^{11}$</td>
<td>$3.74 \times 10^8$</td>
<td>$3.37 \times 10^{18}$</td>
</tr>
<tr>
<td>Saturn</td>
<td>$1.43 \times 10^{12}$</td>
<td>$9.35 \times 10^8$</td>
<td>$3.38 \times 10^{18}$</td>
</tr>
<tr>
<td>Uranus</td>
<td>$2.86 \times 10^{12}$</td>
<td>$2.64 \times 10^9$</td>
<td>$3.36 \times 10^{18}$</td>
</tr>
<tr>
<td>Neptune</td>
<td>$4.52 \times 10^{12}$</td>
<td>$5.22 \times 10^9$</td>
<td>$3.39 \times 10^{18}$</td>
</tr>
<tr>
<td>Pluto</td>
<td>$5.90 \times 10^{12}$</td>
<td>$7.82 \times 10^9$</td>
<td>$3.36 \times 10^{18}$</td>
</tr>
</tbody>
</table>

The period of time required for a planet to complete one cycle around the sun is the circumference of its orbit divided by its speed of travel.

$$T = 2\pi r/v$$ \hfill (7.33)

where $r$ is the distance of the planet from the sun and $v$ is the speed of the planet. We can obtain an expression for the speed of the planet in its orbit by equating the centripetal force to the gravitational force.

$$mv^2/r = GMm/r^2$$ \hfill (7.34)

It follows that the square of the velocity is inversely proportional to the radius of the orbit.

$$v^2 = GM/r$$ \hfill (7.35)

This expression for $v^2$ can be substituted into the square of Equation 7.33 to obtain an equation for the period $T$ in terms of the radius $r$:

$$T^2 = 4\pi^2 r^2/v^2 = 4\pi^2 r^2/(GM/r) = 4\pi^2 r^3/(GM)$$ \hfill (7.36)

This equation can be rearranged to show that the ratio $r^3/T^2$ is a constant, equal in size to the product of the universal gravitational constant times the mass of the sun divided by $4\pi^2$:

$$r^3/T^2 = GM/4\pi^2$$ \hfill (7.37)

Use this expression, the numerical value for $G$, and the data in Table 7.2 to compute the mass of the sun.
ENRICHMENT

7.11 Instantaneous Angular Velocity and Angular Acceleration

We define average angular velocity as \( \omega_{\text{ave}} = \frac{\Delta \Theta}{\Delta t} \). As \( \Delta t \) approaches 0 (\( \Delta t \rightarrow 0 \)), this becomes the instantaneous angular velocity or limiting value of \( \frac{\Delta \Theta}{\Delta t} \) and is written \( \frac{d\Theta}{dt} \). Thus,

\[
\omega_{\text{inst}} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \Theta}{\Delta t} = \frac{d\Theta}{dt}
\]

Similarly

\[
\alpha_{\text{inst}} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}
\]

We can express the torque produced by a force acting through a lever arm \( d \) and making an angle \( \Theta \) with the force \( F \), as follows (see Figure 7.5):

\[
\tau = d \times F \tag{7.38}
\]

This vector notation carries with it the direction of the right-hand rule. The torque direction is perpendicular to the plane defined by \( d \) and \( F \), and in direction a right-hand threaded screw advances when turned from \( d \) to \( F \). This vector product is commonly called the cross product and it has the magnitude of

\[
\tau = d \sin \Theta \ F
\]

where \( d \sin \Theta \) is the effective lever arm of the applied force.

From Newton’s second law we know that

\[
F = \frac{dp}{dt}
\]

and we can write the torque as follows:

\[
\tau = d \times \frac{dp}{dt}
\]

Consider the case of the moving earth around the sun. The gravitational force is parallel to \( d \) and satisfies the centripetal force equation

\[
F_c = \frac{dp}{dt}
\]

Since \( F_c \) is parallel to \( d \) we see that \( \tau = 0 \). Also \( d \times p = d \times mv \), where the magnitude of this expression is given as:

\[
d \times m = m d^2 = I \omega
\]

since \( I = md^2 \) for the earth about the sun at a distance \( d \) and \( \omega \) is the angular velocity of the earth about the sun. The direction of this vector is given by the right-hand rule to be the same as that of \( \omega \). Thus we see that \( d \times p \) is equal to the angular momentum of the earth about the sun. In general we can write the following equation:

\[
L = d \times p = I \omega \tag{7.39}
\]

and it follows that

\[
\tau = \frac{dL}{dt} = d(I \omega)/dt \tag{7.40}
\]

When the torque is zero, as is the case when \( F \) is parallel to \( d \), then \( L \) is a constant and angular momentum is conserved. When the moment of inertia is constant, we find the following equation holds for the motion: \( \tau = I \frac{d\omega}{dt} = I \alpha \)
SUMMARY

Use these questions to evaluate how well you have achieved the goals of this chapter. The answers to these questions are given at the end of this summary with the section number where you can find the related content material.

Definitions

1. The angular displacement of the minute hand of a clock between 12:00 and 12:15 in radians is
   a. $\pi$
   b. $\pi / 2$
   c. $\pi / 3$
   d. $\pi / 15$
   e. $\pi / 4$

2. The angular velocity of the minute hand of a clock in radians per second is
   a. $\pi / 30$
   b. $\pi / 6$
   c. $\pi / 1800$
   d. $\pi / 3600$
   e. $2\pi$

3. If a turntable reaches a final speed of 120 rpm in 2.00 sec, the angular acceleration of the turntable is
   a. 60 rad/sec$^2$
   b. $2\pi$ rad/sec$^2$
   c. 1 rad/sec$^2$
   d. $4\pi$ rad/sec$^2$
   e. $\pi$ rad/sec$^2$

4. The torque produced by tangential force, $F$, applied to the rim of a disk of radius $R$ will be
   a. $F/R$
   b. $F/R^2$
   c. $F\sqrt{R}$
   d. $FR$
   e. $R/F$

5. The angular acceleration produced by the torque in question 4 will be proportional to
   a. $1/I$
   b. $I$
   c. $I/R^2$
   d. $Im$
   e. $F/R$

where $I$ is the moment of inertia of the disk.
6. If a free body is suspended on an axis through its center of mass, you can be sure the body will have center of mass, you can be sure the body will have
   a. clockwise torque
   b. counterclockwise torque
   c. zero weight
   d. zero \( I \)
   e. zero torque

7. The rotational kinetic energy of the mass particles in a rotating disk is
   a. the same for all particles
   b. the greatest for those near axis
   c. greatest for those nearest rim
   d. zero
   e. independent of speed of the disk’s rotation

8. The angular momentum of a system may be
   a. proportional to \( \omega \)
   b. proportional to \( I \)
   c. directed parallel to \( \omega \)
   d. changed in the direction of applied torque
   e. all of these

Equilibrium
9. The conditions necessary for a body to be in static equilibrium are

Rotational Motion and Kinematics
10. An equation that holds for an object of radius \( R \) rolling without slipping at a velocity \( v \) is given as
    a. \( a_t = \alpha R \)
    b. \( v = \omega R \)
    c. \( a_c = g \)
    d. \( a_t = \mu g \)
    e. a and b

11. The angular displacement in time \( t \) for constant angular acceleration is given by
    ____________________________.

12. The angular displacement for an object accelerating at 1 rad/sec\(^2\) from 0 to 4 rad/sec is
    a. 4 rad
    b. 8 rad
    c. 16 rad
    d. 2 rad
    e. zero
Rotational Dynamics

13. It is noted that the maximum angular acceleration produced by a given torque applied to a rod with its axis at its center is two times that produced when the axis is at one end. The ratio of the moments of inertia for $I_{\text{center}} / I_{\text{end}}$ is
a. 2
b. 1/2
c. 4
d. 1/4
e. 8

14. When a chunk of putty of mass M is dropped onto the outer edge (a distance R from the axis) of a turntable, the angular velocity of the turntable is reduced by half. The moment of inertia of the turntable must be ____________________.  

Equilibrium Problems

15. A uniform board is supported by a pivot one-fourth its length from one end. Find the force applied to the opposite end necessary to balance the board. Find the force on the pivot. (M = mass of board and L = length of the board.)

Answers
1. b (Section 7.2)  5. a (Section 7.5)  9. $\Sigma F = 0, \Sigma r = 0$
2. c (Section 7.2)  6. e (Section 7.4) (Section 7.4)  13. b (Section 7.5)
3. b (Section 7.2)  7. c (Section 7.7)  10. e (Section 7.2)  14. $MR^2$ (Section 7.8)
4. d (Section 7.3)  8. e (Section 7.8)  11. $\Theta = \omega_o t + 1/2 \alpha t^2$ (Section 7.4)
   (Section 7.2)  12. b (Section 7.2)

ALGORITHMIC PROBLEMS

Listed below are the important equations from this chapter. The problems following the equations will help you learn to translate words into equations and to solve single concept problems.

Equations

\[ \omega_{\text{ave}} = (\Theta_2 - \Theta_1) / (t_2 - t_1) = \Delta \Theta / \Delta t \] (7.1)
\[ \alpha_{\text{ave}} = (\omega_2 - \omega_1) / (t_2 - t_1) = \Delta \omega / \Delta t \] (7.3)
\[ \omega_f = \omega_o + \alpha t \] (7.4)
\[ \omega_{\text{ave}} = (\omega_f + \omega_o) / 2 \] (7.5)
\[ \Delta \Theta = \Theta_f - \Theta_o = \omega_{\text{ave}} t = [(\omega_f + \omega_o) / 2] t = \omega_o t + 1/2 \alpha t^2 \] (7.6)
\[ 2\alpha (\Theta_f - \Theta_o) = \omega_f^2 - \omega_o^2 \] (7.7)
\[ \tau = F d \] (7.18)
\[ \tau = I \alpha \] (7.20)
\[ I = \sum_{i=1}^{n} m_i r_i^2 \quad (7.22) \]

\[ \text{Work} = \tau \cdot \Theta \quad (7.24) \]

\[ KE_{\text{rot}} = (\frac{1}{2}) I \omega^2 \quad (7.26) \]

\[ L = I \omega \quad (7.29) \]

**Problems**

1. A turntable turns through 10.0 revolutions in 5.00 sec. Find the angular velocity of the turntable in rad/sec.

2. A phonograph turntable is rotating at 45.0 rpm. Find the angle that a point on the rim of the turntable turns through in 12.0 sec.

3. A turntable starting from rest reaches an angular velocity of 10.0 rad/sec in 4.00 sec. Find the angular acceleration of the turntable.

4. If you can exert a 20.0-N force on a wrench 25.0 cm long, find the maximum torque that can be exerted by using this force and wrench.

5. A torque of 10.0 N-m can be applied to a flywheel that has a moment of inertia of 5.00 kg\(\cdot\)m\(^2\). Find the angular acceleration of the flywheel that is produced by this torque.

6. Find the rotational kinetic energy of the flywheel in problem 5 if its angular velocity is 100 rad/sec.

**Answers**

1. 4\(\pi\) rad/sec
2. 9 rev or 18 \(\pi\) rad
3. 2.50 rad/sec
4. 5.00 N-m
5. 2.00 rad/sec
6. 25,000 J.

**EXERCISES**

These exercises are designed to help you apply the ideas of a section to physical situations. When appropriate the numerical answer is given at the end of each exercise.

**Section 7.2**

1. What is the average angular velocity of the earth spinning about its axis? What is the average angular velocity of the earth in its orbit about the sun? [7.27 \(\times\) 10\(^{-5}\) rad/sec, 1.99 \(\times\) 10\(^{-7}\) rad/sec]

2. A wheel is fitted with two electrical contacts, separated by 10° for timing circuits. What angular velocity (in rad/sec and rpm) must the wheel have for a time interval between these two contacts for a. 0.200 sec? b. 0.00400 sec? [a. 0.873 rad/sec, 8.33 rpm; b. 4.36 \(\times\) 10\(^{1}\) rad/sec, 4.17 \(\times\) 10\(^{1}\) rpm]

3. An automobile is traveling on a level road at 80.0 km/hr. The diameter of the wheels is 0.750 m. What is the angular velocity of the wheel about its spindle? [59.2 rad/sec]
4. If the car in exercise 3 stops in a distance of 50.0 m on the highway,
a. how many revolutions did the wheel make?
b. what is the average angular acceleration of the wheel in stopping? [a. 21.2 rev; b. 13.1 rad/sec²]

Section 7.4
5. A dentist grips a pair of forceps exerting a force of 20.0 N on each side at a distance of 10.0 cm from the pivot. The tooth is gripped at a point of 0.500 cm from the pivot. What is the force exerted on the tooth? [400 N on each side]
6. The radius of the steering wheel of a car is 25.0 cm. The driver exerts a force of 10.0 N tangent to the rim of the steering wheel. What is the torque acting to produce rotation? [2.50 N-m]
7. The weight on the front wheels of an automobile is 900 N, and the weight on the rear wheels is 7200 N. The wheels are 2.80 m apart. Where is the center of gravity? [2.48 m from front]
8. A 4.00-m uniform plank is resting on a pier with 1.50 m extending beyond the edge of the pier. The plank weighs 1400 N. How far can a 50.0-kg student walk beyond the edge of the pier before the plank will tip? [1.43 m]

Section 7.5
9. Find the torque necessary to give a 2.00-kg disk of 10.0 cm radius an angular acceleration of 0.100 rad/sec². [1.00 x 10⁻³ N-m]
10. A rope wrapped around a pulley (disk) of mass \( M \) and radius \( R \) is given an angular acceleration by an equal mass hanging on the end of the rope. Find the angular acceleration. [2/3 \( g/R \)]

Section 7.7
11. Compare the rotational kinetic energies of a hoop and a sphere of equal mass and radius traveling at the same center of mass speed. [KE sphere / KE hoop = 0.4]

Section 7.8
12. A tubular ring (mass = \( M \)) of radius \( R \) is filled with a fluid of mass (\( M/4 \)). If the ring, filled with fluid, spins about an axis through its center at 1.00 rpm, find the spin rate after all the fluid leaks out. [1.25 rpm]

Section 7.9
13. Find the necessary work to roll a ball of mass \( M \), in kilograms, and radius \( R \), in meters, up an incline. Assume the ball starts from rest and has a speed of \( v \) meters/sec at the incline height of \( H \) meters. \[ MgH + \frac{7}{10} mv^2 \]

Section 7.10
14. Using 365 days as Earth’s period and its solar orbit radius as 1.00 astronomical unit (AU), find the distance of an asteroid from the sun that has a period of 2920 days. [4.0 AU]
PROBLEMS

Each problem may involve more than one physical concept. The answer is given at the end of the problem. A problem that requires material from the Enrichment Section, 7.11, is marked by a dagger (†).

15. A load of 200 N is supported by a crane as shown in Figure 7.21. The boom AB is uniform and weighs 20.0 N. What is the tension in the tie BC? What is the magnitude and direction of the force exerted by the support at A on the boom? [126 N, 191 N, 48.8° above horizontal]

16. A man holds a 30-N weight in his hand, keeping his forearm at rest in a horizontal position. In Figure 7.22, E is the elbow, H is the hand, and F is the force exerted by the muscle at M which makes an angle of 60° with the horizontal. What is the magnitude of F and the force acting at E? [208 N, 150 N down at an angle 55.3° below horizontal]

17. A man carries a uniform pole 2.40 m long, and weighing 50.0 N over his shoulder, holding it horizontally. He pulls down with his hand at the front end 0.600 m from his shoulder. A 20.0-N bundle hangs from a point 0.100 m from the other end. What is the force exerted by the man’s hand and the force exerted against his shoulder? [107 N, 177 N]

18. A uniform ladder 5.00 m long, weighing 100 N leans against a smooth vertical wall. The distance of the foot of the ladder from the wall is 3.00 m. What is the force exerted by the wall on the ladder when an 800-N man stands 1.00 m from the upper end of the ladder? What is the magnitude and direction of the force of the ground on the ladder? What is the coefficient of friction between the ground and ladder for this case? [518 N, 1040 N, 60.1° above the horizontal, 0.576]

19. Find the center of gravity of a unit that is made of two lengths of uniform circular rod (see Figure 7.23). [12 cm to right of intersection, 4.0 cm below intersection]
20. Find the center of gravity of a sheet of plastic of uniform thickness with dimensions as shown in (Figure 7.24).

[7 cm to the right and 3 cm above the lower left corner]

21. A hoop of mass m and radius r is supported by a knife edge (Figure 7.25). If displaced from equilibrium position, it will oscillate about the equilibrium position. What is the moment of inertia of the hoop about the knife edge? What is the torque tending to return it to equilibrium if it is displaced an angle Θ from the equilibrium position? [2 Mr², mgr sinΘ]

22. In sharpening a paring knife, a man uses a hand-driven emery wheel which has a handle length of 10.0 cm. Assume a force of 40.0 N is exerted in a direction perpendicular to the handle and that 200 revolutions are made in the grinding process. How much work did he do in sharpening the knife? [5030 J]

23. A solid cylinder is mounted on a frictionless horizontal axis. The cylinder has a radius of 10.0 cm and a mass of 4.00 kg. A constant force of 10.0 N is applied to a cord wrapped around the cylinder. What is the angular acceleration? Would the acceleration be the same if a 10.0-N weight were attached to the cord? [50.0 rad/sec², no]

24. A pulley is mounted on fixed ball bearings (frictionless), and a 5.00-kg mass is hanging from a cord wrapped around the hub which has a radius of 4.00 cm (Figure 7.26). The mass has a downward acceleration of 3.80 m/sec². What is the moment of inertia of the pulley? If the mass of the pulley is 6.00 kg, what is the radius of a hoop which has the same moment of inertia? [.0126 kg m², 0.0458 m]

25. A sphere rolls down an inclined plane that has an elevation of 1.00 m and length of 2.00 m. What is its velocity at the foot of the incline? Assume no frictional losses. How would this velocity compare with the velocity of a body sliding down an equivalent frictionless incline. Compare the kinetic energies of translation and rotation.

[3.74 m/sec, 4.43 m/sec, KE_{rot} = 2/7 KE_{tot}, KE_{trans} = 5/7 KE_{tot}]

26. A merry-go-round is moving with an angular velocity of 2.00 rad/sec. Assume the merry-go-round has a moment of inertia of 360 kg-m² and a radius of 2.00 m. A 30.0-kg boy who was riding on the merry-go-round decides to walk on a support toward the renter of rotation. What is the angular velocity of the merry-go-round when he reaches 0.500 m from the center? Neglect frictional effects and treat the boy as a point mass. [2.61 rad/sec]

27. Given the schematic of the forearm holding a mass $M$, shown in Figure 7.27, find the force exerted by the biceps $F$ on the forearm for equilibrium if the mass of the forearm is $M/10$. Assume that the center of mass of the forearm is 16.0 cm from $E$. [8.4 $Mg$ N]

28. Figure 7.28 shows a person during push ups. The mass of the person is 80.0 kg. Find the force exerted by each hand and each foot on the floor. [231 N, 161 N]

29. A crouching man on tiptoes puts considerable tension on the Achilles tendon as shown in Figure 7.29. Find the tension on the tendon for a 100-kg man (one-half of his weight is supported by each foot) using the data given. [980 N]

30. The biceps muscle system is shown in Figure 7.30 with typical dimensions. What is the force in the biceps muscle if 2.00 kg is supported by the hand in the horizontal position?
   a. if $\alpha = 80^\circ$?
   b. if $\alpha = 30^\circ$? [a. 160 N; b. 314 N]
31. A 70.0-kg man puts all of his weight on the ball of his right foot. What is the tension of the leg muscle? See Figure 7.31 for relevant data. [2740 N]

![Figure 7.31 Problem 31.](image)

32. Suppose that you are standing on your right leg and holding a vertical cane in your left hand which supports 1/6 of your weight and is 30 cm to the left of the center of mass of your body. Where will your right foot now be placed? See Figure 7.13. With your foot in the position indicated above, what will be the new value of $F$? Remember the location of the center of gravity and the length of your leg will remain the same. Compare your values of $F$ with and without a cane. [6 cm to right of your center of mass, 360 N, $F_{\text{without cane}} / F_{\text{cane}} = 2.66$]

33. A clever inventor at the time of Newton decided that he could put carriages in orbit around the earth by releasing them horizontally from the top of a ferris wheel of 30.0-m radius. Find the necessary angular velocity of the ferris wheel to obtain the orbital condition for the carriages. [17.2 rad/sec]

34. There are two different ways you can lift a load of weight $L$ shown in Figure 7.32a and b. Assume the center of mass of your upper trunk is two-thirds of the distance $r$ from your hip pivot to your shoulder, that the load is twice the weight $W$ of your upper trunk, and that the back muscle can be represented by the simple forces $M$ and $N$ as shown, where $r$ is about 20 cm. Compare the forces on your spine in the two cases. [$M$ in part a about 10 times larger than in part b]

35. If a person turns his head through an angle of $60^\circ$ in 0.10 sec, what torque must be applied by the neck muscles? Assume that the head has a mass of 5 kg and is a uniform sphere with a radius of 8 cm. The axis of rotation passes through the center of mass and the motion starts from rest and has constant angular acceleration. [2.70 N-m]

$\dagger$36. The angular displacement of a rotating body is given by $\Theta = 4t^2 - 16t + 6$, with $\Theta$ in radians and $t$ in seconds. What is
a. the angular velocity at $t = 1$ sec, at $t = 3$ sec?
b. the angular acceleration?
c. the time the body is at rest?
d. the angular displacement when at rest? [-8 rad/sec, 8 rad/sec$^2$, 2 sec, 6 rad]
†37. A 2 kg particle moves through the position $x = 5$ m, at a velocity of $4$ m/sec in a direction $37^\circ$ above the $+x$ axis. Find its angular momentum relative to the origin. 

[24 kg m$^2$/sec along the $+z$ axis]