Chapter 15
SIMPLE HARMONIC MOTION

GOALS

When you have mastered the contents of this chapter, you will be able to achieve the following goals:

Definitions

Define each of the following terms, and use it in an operational definition:

period
simple harmonic motion
amplitude
phase angle
frequency
restoring force
damping

UCM and SHM
Correlate uniform circular motion and simple harmonic motion.

SHM Problems
Solve problems involving simple harmonic motion.

Energy Transformations
Analyze the transfer of energy in simple harmonic motion.

Superposition
Explain the application of the principle of superposition to simple harmonic motion.

Natural Frequencies
Calculate the natural frequencies of solids from their elastic moduli and density values.

PREREQUISITES

Before beginning this chapter you should have achieved the goals of Chapter 5, ***Energy***, Chapter 7, ***Rotational Motion***, and Chapter 13, ***Elastic Properties of Materials***.

Chapter 15
SIMPLE HARMONIC MOTION

15.1 Introduction

You are familiar with many examples of repeated motion in your daily life. If an object returns to its original position a number of times, we call its motion repetitive. Typical examples of repetitive motion of the human body are heartbeat and breathing. Many objects move in a repetitive way, a swing, a rocking chair, and a clock pendulum, for example. Probably the first understanding the ancients had of repetitive motion grew out of their observations of the motion of the sun and the phases of the moon.

Strings undergoing repetitive motion are the physical basis of all stringed musical instruments. What are the common properties of these diverse examples of repetitive motion?

In this chapter we will discuss the physical characteristics of repetitive motion, and we will develop techniques that can be used to analyze this motion quantitatively.

15.2 Kinematics of Simple Harmonic Motion

One common characteristic of the motions of the heartbeat, clock pendulum, violin string, and the rotating phonograph turntable is that each motion has a well-defined time interval for each complete cycle of its motion. Any motion that repeats itself with equal time intervals is called *periodic motion*. Its *period* is the time required for one cycle of the motion.

Let us analyze the periodic motion of the turntable of a phonograph. Suppose that we place a marker on a turntable that is rotating about a vertical axis at a uniform rate in a counterclockwise direction when viewed from above. If you observe the motion of the marker in a horizontal plane-that is, viewing the turntable edge-on-the marker will seem to be moving back and forth along a line. The motion you see is the projection of uniform circular motion onto a diameter and is called *simple harmonic motion* (Figure 15.1). To derive the equation for simple harmonic motion, project the motion of the marker upon the diameter AB . The displacement is given relative to the center of the path O and is represented by $x = OC$. From Figure 15.1 we see that $x = R\cos\theta$, where R is the distance of the marker from the axis of rotation. The maximum displacement of the motion is called the *amplitude* of the motion and is represented by the symbol A.

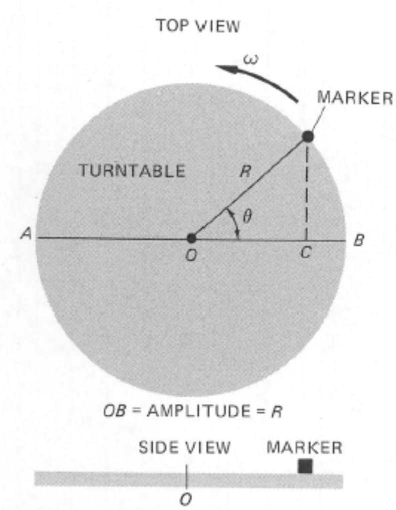


FIGURE 15.1
Simple harmonic motion. Projection of uniform circular motion upon a diameter.

The displacement of the marker in a direction parallel to the diameter *AOB* is then given by the following equation,

$$x = A \cos\theta \qquad (15.1)$$

where θ is the angle through which the marker on the turntable has turned. Since we know that the turntable is rotating with a constant angular velocity ω , we recall from Chapter 7 that we can write an expression for the angular displacement θ as the angular speed times the time plus the starting angle,

$$\theta = \omega t + \phi$$

where t is the time of rotation and ϕ, the *phase angle*, is the angular displacement at the beginning, $t = 0$. If we choose the starting position along the line *DOB,* then $\phi = 0$ at $t = 0$. In general, the equation for the x-displacement is given by

$$x = A \cos(\omega t + \phi) \qquad (15.2)$$

The velocity of the marker for the position shown in Figure 15.1 is tangential to the circle of motion of the marker and in the direction shown in Figure 15.2 . You may recall from Chapter 7 that the magnitude of the velocity is given by

$$v = 2\pi R n = \omega R = \omega A \qquad (7.12)$$

where n is the number of revolutions of the turntable in one second, ω is the angular speed in radians per second, and R is the radius of the circle of motion and is equal to the amplitude of displacement A .

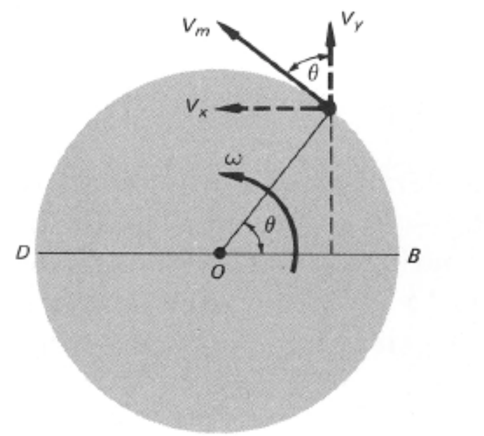


FIGURE 15.2
Velocity in simple harmonic motion. Projection of the velocity of uniform circular motion upon a diameter.

EXAMPLE

What is the velocity of a point on the rim of the standard 12-inch long-playing phonograph record?

$R = 15.2$ *cm*

$\omega = 33\ (1/3)$ rpm $= 5.56 \times 10^{-1}$ rev/sec $= 3.49$ rad/sec.

$v = \omega R = 53.1$ cm/sec

The velocity of the marker in a direction parallel to the line *DOB* is shown by the component $v x$ in Figure 15.2 , where

$$v_x = -v \sin\theta = -\omega A \sin\theta \qquad (15.3)$$

The negative sign indicates that the direction of motion is in the negative x-direction. When $\sin\theta$ is positive, the velocity v_x is in the negative x-direction. Notice $\sin\theta$ is always positive for $0 \le \theta \le 180^{-}$; so v_x is negative for those angles and positive for the rest of the motion of one cycle.

The acceleration of the marker for the position shown in Figure 15.1 is perpendicular to the velocity, toward the center of circular motion as shown in Figure 15.3 . The magnitude of the acceleration was derived in Chapter 3, $a = v2/ R$ (Equation 3.21). for the case we are considering, we can write the acceleration in terms of the angular speed and the amplitude,

$$a = v^2/ R = \omega^2 R^{2}/R = \omega^2 R = \omega^2 A \qquad (15.4)$$

The projection of the acceleration vector onto the line *DOB* is shown by *ax* in Figure 15.3 where

$$a_x = -\omega^2 A \cos\theta \qquad (15.5)$$

The negative sign indicates that the acceleration is in the negative x-direction. The $\cos\theta$ is positive for $-90^{\circ} \le \theta \le 90^{\circ}$; so a_x is in the negative x-direction for these angles and positive for the rest of each cycle of motion. Notice that from Equation 15.1 we know that $A \cos\theta$ is the x-displacement.

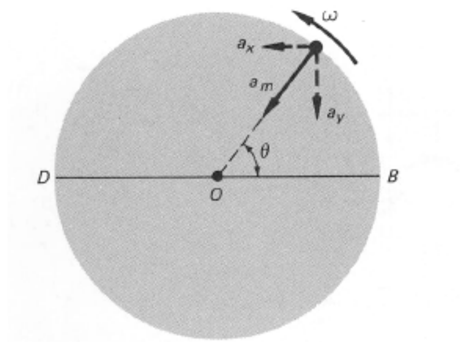


FIGURE 15.3
Acceleration in simple harmonic motion. Projection of the acceleration of uniform circular motion upon a diameter.

The equation for the acceleration can be rewritten in terms of the x-displacement

$$a_x = -\omega^2 x \qquad (15.6)$$

If we substitute the equation for the angular displacement as a function of time, $\theta = \omega t + \phi$, into the equations for x-displacement, x-velocity, and x-acceleration, then the linear displacement, velocity, and acceleration are given in general by the following equations:

$$x = A \cos(\omega t + \phi) \qquad (15.7)$$

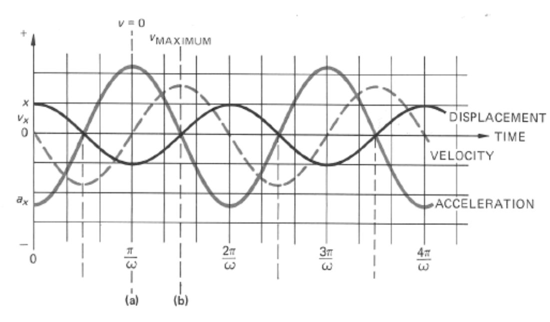
$$v_x = -\omega A \sin(\omega t + \phi) \qquad (15.7)$$

$$a_x = -\omega^2 A \cos(\omega t + \phi) \qquad (15.7)$$

Assume $x = A$ at time $t = 0$. Then $\phi = 0$. Then these equations can be represented in graphical form as shown in Figure 15.4 .

The curves in Figure 15.4 show that at the time of zero velocity Figure 15.4a , the acceleration and the displacement are maximum. At a time of maximum velocity Figure 15.4b , the acceleration and the displacement are zero.

FIGURE 15.4
Displacement, velocity, and acceleration of simple harmonic motion as functions of time.



For simple harmonic motion the acceleration is proportional to the displacement x and is oppositely directed (Equation 15.6). If the displacement is to the right of the equilibrium position, then the acceleration is to the left, and vice versa. The angular speed ω is a constant, a characteristic of the motion. The angular speed can be expressed in terms of the frequency, or the period, of the motion. In uniform circular motion we defined the number of revolutions per second as the *frequency f* . Then

$$\omega = 2\pi n = 2\pi f \tag{15.8}$$

where f is the number of cycles per second (hertz). The period T of one vibration is equal to the reciprocal of f.

$$T = 1/f \tag{15.9}$$

Then the angular speed is proportional to the reciprocal of the period of the motion,

$$\omega = 2\pi/T \tag{15.10}$$

where T is the period of the motion. By substituting the expression for ω from Equation 15.6 into Equation 15.8 and solving for f , we get the relationship between the frequency, the acceleration and the displacement,

$$f = 1/(2\pi) \text{ SQR RT } [- a/x] = 1/T \text{ or } T = 2\pi \text{ SQR RT } [-x/a] \tag{15.11}$$

The value of $- a/x$ is always positive because a and x are in opposite directions. The above equation can be used to calculate either the frequency (period) or the acceleration or the displacement, if you know the other two variables.

EXAMPLES

1. A butcher throws a cut of beef on spring scales which oscillates about the equilibrium position with a period of $T = 0.500$ s. The amplitude of the vibration is $A = 2.00$ cm (path length 4.00 cm). Find:

   a. frequency
   b. the maximum acceleration
   c. the maximum velocity
   d. the acceleration when the displacement is 1.00 cm
   e. the velocity when the displacement is 1.00 cm
   f. the equation of motion as a function of time if the displacement is A at $t = 0$

SOLUTIONS

a. The frequency $f = 1/T = 1/0.500 = 2.00$ hertz (vibrations/s) according to Equation 15.9.

b. By Equation 15.8, the angular velocity is given by

$$\omega = 2\pi f = 4.00\, \pi \text{ rad/sec} = 12.6 \text{ rad/s}$$

Then by Equation 15.7

$$a_x = -\omega^2 A \cos(\omega t + \phi)$$

The maximum acceleration occurs when $A \cos(\omega t + \phi)$ is equal to $-A$, or -2.00 for this problem.

$$a_{max} = -(4.00\, \pi)2\, (-2.00) = +\, 32.0\, \pi^2 \text{ cm/s2} = 316 \text{ cm/s}^2$$

c. The velocity is given by Equation 15.7:

$v = -\omega A \sin(\omega t + \phi)$

The velocity will be maximum when $A \sin(\omega t + \phi)$ is equal to $-A$; so

$v_{max} = \omega A = 4.00\pi \times 2.00 = 8.00\pi$cm/s = 25.1 cm/s

d. The acceleration is given by

$a = -\omega^2 x$

which is Equation 15.6. When the displacement is 1.00 cm,

$a = -(4.00\pi)^2 \times 1.00 = -16.0\pi^2 = 158$ cm/s^2

e. Use Equation 15.2 to find the angular displacement $(\omega t + \phi)$, and then use Equation 15.7 to find the velocity,

$\cos(\omega t + \phi) = x/A = 0.500$

So the $\sin(\omega t + \phi)$ = SQR RT [3/2] = 0.866

$v = -\omega A \sin(\omega t + \phi) = -4.00\pi \times 2.00 \times 0.866$

Therefore, the velocity when the displacement is 1 cm is

$v = 6.93\ \pi$cm/s = -21.8 cm/s

f. $x = A \cos(\omega t + \phi)$

by Equation 15.2. If $x = A$ at time $t = 0$, then $\phi = 0$. So $x = 2.00 \cos 4.00\pi t$ cm.

2. In a system undergoing simple harmonic motion, the acceleration is -20.0 cm/s^2 for a displacement of 5.00 cm. What is the frequency and period of motion? What other information is needed to write an equation of motion? Let $x = 5.00$ cm, and $a = -20.0$ cm/s^2. Then $T = 2\pi$ SQR RT [-x/a] (Equation 15.11). Therefore, the period of motion is$T = 2\pi$ SQR RT [+5.00/20.0] = 3.14 s, and the frequency is $f = 1/T = 1/3.14$ sec = 0.318 Hz. If you know the location at any time, then you can determine ϕ and can write the exact equation of the motion.

15.3 Dynamics of Simple Harmonic Motion

We have considered simple harmonic motion without regard to the forces that produce such motion. We now want to discover the common characteristics of the forces that produce simple harmonic motion (SHM). In many cases, the recognition of this SHM force not only allows you to predict harmonic motion, but it allows you to predict the frequency of the motion.

What kind of forces produce simple harmonic motion? Look at Equation 15.6. According to this equation, the acceleration in a SHM system must be proportional to the displacement of the system from equilibrium and in the opposite direction. Let us combine this equation with our knowledge of Newton's second law

$F_x = ma_x$ where $a_x = -\omega x$ (4.1)

so $F_x = -m\omega^2 x$

according to Equation 15.6, where the mass m and the angular speed ω are constants of the system. The component of the force F_x in the direction of motion is a *restoring force*, acting opposite in direction to the displacement as indicated by the negative sign. The magnitude of F_x is proportional to the displacement; F_x is a Hooke's law force. Therefore, Hooke's law force systems (Section 13.3) produce simple harmonic motion. If the Hooke's law force equation is written as follows, $F = -kx$ (Equation 13.3) then the force constant k is equal to $m\omega^2$. We can derive equations for the period and frequency of the simple harmonic motion that arises from a Hooke's law force by making use of the equality between the ratio of the force constant to the mass of the system and the square of the angular speed

$$k/m = \omega^2 = (2\pi f)^2 = (2\pi/T)^2 \qquad (15.7)$$

frequency = 1/2π SQR RT [k/m] Hz from above and (15.8)

period = 2π SQR RT [m/k] sec (15.9)

In summary, all Hooke's law force systems will produce simple harmonic motion. Any system of simple harmonic motion can be used to deduce a Hooke's law force. There are many examples of Hooke's law systems such as spring balances and simple pendulums. In fact, in the enrichment section we will use Taylor's theorem to show that almost *all* equilibrium systems exhibit simple harmonic motion near equilibrium.

EXAMPLE

It is known that a load with a mass of 200 g will stretch a spring 10.0 cm. The spring is then stretched an additional 5.00 cm and released. Find:

a. the spring constant
b. the period of vibration and frequency
c. the maximum acceleration
d. the velocity through equilibrium positions
e. the equation of motion

SOLUTIONS

a. If the force acting $mg = 0.200 \times 9.80$ N and $x = 0.10$ m, then the spring constant is

$k = F/x = 0.200 \times 9.80/0.10 = 19.60$ N/m

b. To find the period of vibration let T = 2π SQR RT [m/k]. Then,

T/m = 2πSQR RT [0.200/19.6] = 2π/7 SQR RT [2] = SQR RT [2]/7 π = 0.634 s

The frequency is

$f = 1/T = 1/0.634 = 1.59$ Hz

c. Given that the amplitude = 5.00 cm,

$\omega^2 = k/m = (19.6 \text{ N/m})/0.200 \text{ kg} = 98.0/\text{s}^2$

Then the maximum acceleration is

$| a_{max} | = | -\omega^2 A | - 98.0 \times .0500 | = 4.90 \text{ m/s}^2$

d. To find the velocity through the equilibrium position let $|v_{max}| = A\omega$ = (0.0500 m) (SQR RT [98.0/s^2] = 0.495 m/s

e. A = 5.00 cm, ω = SQR RT [98]/sec = 9.90 sec-1, and $\phi = 0$, since $x = A$ at time $t = 0$. Therefore, $x = 5.00 \cos (9.90t)$ cm

15.4 Energy Relationships in Simple Harmonic Motion

You will recall that in previous chapters we have been able to use the concept of conservation of energy to solve mechanical problems. The conservation of energy has been especially useful in problems that involve systems whose total energy is a constant. Perhaps you wonder if energy analyses of SHM systems will yield worthwhile results. Let us consider the energy relationships in simple harmonic motion.

We can show that the potential energy associated with a $-kx$ force is equal to (½) kx^2 for a displacement of x. Since potential energy is equal to the work,

$$\text{PE} = W = (F_{\text{ave}})(x) = (kx)\,x/2 = (½)kx^2 \qquad (15.12)$$

The total energy of the system is equal to the sum of the potential and kinetic energy,

$$\text{Energy} = \text{KE} + \text{PE} = (½)mv^2 + (½)\,kx^2 \qquad (15.13)$$

where v is the velocity of the moving object whose mass is m. We can use the equations for displacement and velocity as functions of time (Equations 15.2 and 15.7) to write an expression for the total energy as a function of time:

$$E = (½)m\,[\,-\,\omega A \sin(\omega\, t + \phi)\,]^2 + (½)\,k\,[\omega \cos(\omega\, t + \phi)\,]^2$$

$$E = (½)m^2\omega A^2 \sin^2(\omega t + \phi) + (½)kA^2 \cos^2(\,\omega t + \phi)$$

where ω2 $=k/m$ for Hooke's law system. Therefore,

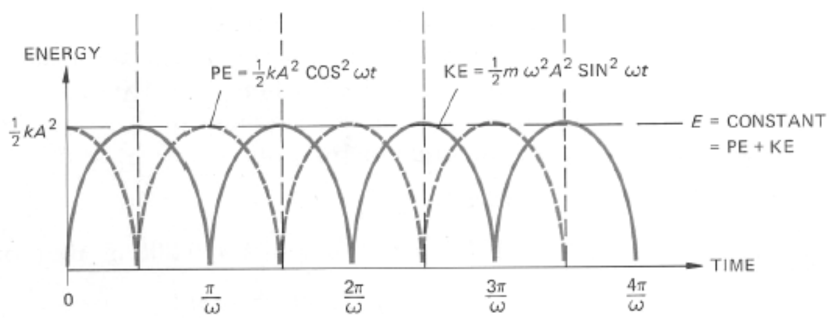
$$E = (½)kA^2\,[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

From our knowledge of trigonometry we know that for any angle θ, $\sin^2\theta + \cos^2\theta = 1$,

$$E = (½)kA^2 \qquad (15.14)$$

The total energy of a SHM system is a constant, independent of time, and equal to one-half of the product of the force constant and the square of the amplitude of oscillation. The relationship between the potential energy and the kinetic energy as a function of time is shown in Figure 15.5.

FIGURE 15.5
Plot of the kinetic energy and the potential energy of SHM as a function of displacement. Note that the sum of KE and PE for any displacement is a constant: total energy = KE + PE = constant.



When the potential energy has its maximum value (½)kA^2, the kinetic energy is zero, and the object is instantaneously at rest at its position of maximum displacement, $x = A$. When the kinetic energy has its maximum value (½)$m\omega^2A^2$ or (½)kA^2, the potential energy is zero, and the object is moving with its maximum velocity at zero displacement. At all other positions neither the potential energy nor kinetic energy is zero. Since the total energy is a constant equal to (½)kA^2, we can use Equation 15.13 to find an expression for the velocity as a function of displacement,

$(½)kA^2 = (½)kx^2 + (½)mv^2$

which we use to solve for velocity at any displacement x,

$v^2 = k/m\ (A^2 - x^2)$ (15.15)

Remember that k/m is equal to ω2; so the magnitude of the velocity is given by

$v = \omega$ SQR RT $[A^2 - x^2]$ (15.16)

The maximum velocity occurs as the object passes through its equilibrium position, $x = 0$,

$v_{max} = \pm\omega A$ (15.17)

This is the same result we derived earlier from Equation 15.7.

EXAMPLE

A 0.500-kg mass is vibrating in a system in which the restoring constant is 100 N/m; the amplitude of vibration is 0.200 m. Find

a. the energy of the system
b. the maximum kinetic energy and maximum velocity
c. the PE and KE when $x = 0.100$ m
d. the maximum acceleration
e. the equation of motion if $x = A$ at $t = 0$

SOLUTIONS

a. If $k = 100$ N/m and $A = 0.200$ m, then the total energy of the system = $(½)kA^2$

$E = (½)\ (100)(0.200)2 = 2.00$ J

b. Maximum KE = energy of system = 2.00 J. To find the maximum velocity let

KE = $(½)mv_{max}{}^2 = (½)\ (0.500)v_{max}{}^2 = 2.00$ J

Then $v_{max}{}^2 = 8.00$; so, $v_{max} = 2.83$ m/s

c. The total energy = $(½)kx^2 + (½)mv^2$. At $x = 0.100$,

PE = $(½)\ (100)(0.100)^2 = 0.500$ J

and

KE = total energy - PE = 2.00 - 0.500 = 1.50 J

d. The maximum acceleration $a_{max} = |\ -\omega^2 A\ |$ and $\omega^2 = 100/0.500 = 200\ s^{-2}$.

$a_{max} = (200)(0.200) = 40.0\ m/s^2$

e. If $x = A$ and $t = 0$, then $\phi = 0$. Using the equation $x = A\ (\cos \omega t + \phi)$ where ω = SQR RT $[k/m]$; ω = SQR RT [200] = 14.1 rad/s, $x = 0.200 \cos 14.1\ t$ m.

In summary, the universality and the simplicity of simple harmonic motion is very appealing. The total energy of a SHM system is conserved. From the dynamics of the system, that is, the force constant and mass, all of the properties of the system can be calculated if the maximum displacement is known. From the kinematics of the system, the angular speed, and the amplitude, all the properties of the system can be calculated if the mass is known. Then if the position of the object is known at any time, the system is completely determined. We have summarized the relationships for a SHM system in Table 15.1 .

TABLE 15.1
Summary Table for Simple Harmonic Motion

Physical Quantity	Expressed in Terms of Displacement x	Expressed in Terms of Time t
Displacement (x)		$x = A\cos(\omega t + \phi)$
Velocity (v)	$v = \pm\sqrt{k/m}\sqrt{A^2 - x^2} = \omega\sqrt{A^2 - x^2}$	$v = -\omega A\sin(\omega t + \phi)$
Acceleration (a)	$a = -\left(\frac{k}{m}\right)x = -\omega^2 x$	$a = -\omega^2 A\cos(\omega t + \phi)$
Restoring force (F_R)	$F_R = -kx$	$F_R = -kA\cos(\omega t + \phi)$
Potential energy	$PE = \frac{1}{2}kx^2$	$PE = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$
Kinetic energy	$KE = \frac{1}{2}k(A^2 - x^2)$	$KE = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$
Frequency (f)	$f = 1/2\pi\left(\sqrt{-\frac{a}{x}}\right) = 1/2\pi\left(\sqrt{\frac{k}{m}}\right)$	$f = \omega/2\pi$
Period (T)	$T = 2\pi\sqrt{-x/a} = 2\pi\sqrt{m/k}$	$T = 2\pi/\omega$

15.5 Damping, Natural Frequencies, and Resonance

In real systems simple harmonic motion is subject to energy loss due to friction. This energy loss process is called *damping*, and it is characterized by decreasing amplitude as shown in Figure 15.6. The degree of damping may be classed as underdamped (curve A), critically damped (curve B), or overdamped (curve C) as illustrated in Figure 15.7. The critically damped case corresponds to the most rapid return to equilibrium for a damped system. Critical damping is the desired situation for such systems as car suspensions, motor mounts on equipment, and meter movement in electrical instruments.

FIGURE 15.6
Simple harmonic motion showing damping.

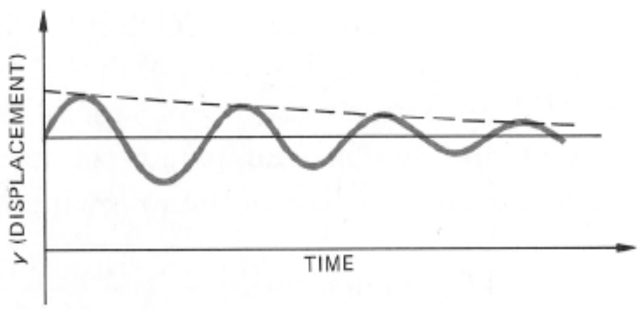
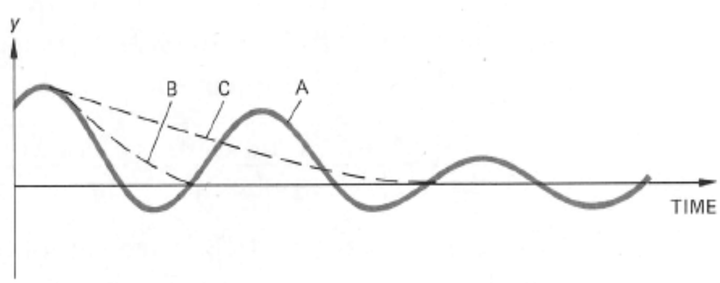
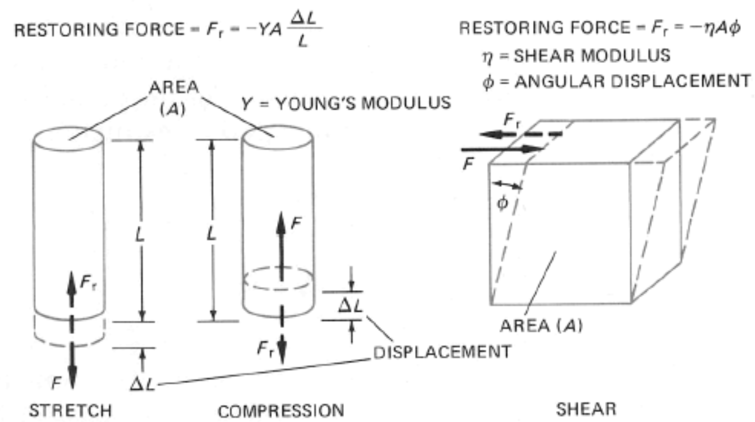


FIGURE 15.7
The degrees of damping: (a) underdamped, (b) critically damped, and (c) overdamped SHM.



A Hooke's law model for materials means that the molecular restoring forces in the material are linear. We might then expect a material to oscillate at some frequency if we can displace its molecules. Three common displacements for solids, the longitudinal stretch, the longitudinal compression, and the shear stress are shown in Figure 15.8. The displacement forces can be related to the elastic constants of the material (see Chapter 13).

FIGURE 15.8
Stretch, compression, and shear displacements in solids.



For each of these displacements our understanding of SHM predicts a *natural frequency* that is determined by the force constant and the inertial parameter for the displacement involved. We write Equation 15.8 in the following form:

natural frequency = f_o =1/(2π)SQR RT[restoring force constant/inertial parameter] (15.18)

If we excite a compressional vibration in a solid of cross-sectional area A, the natural frequency will be related to Young's modulus and the linear density of the material,

f_o = 1/(2π) SQR RT [Young's modulus/((density)(area))]

= 1/(2π) SQR RT [Young's modulus/linear density] (15.19)

From Tables 8.1 and 13.1, we can obtain the values of density, Young's modulus, and the shear modulus of some materials. From these data we can calculate the natural frequencies of special configurations of these materials.

EXAMPLE

Using the data from Tables 8.1 and 13.1, find the ratio of the natural frequency of bone to that aluminum for samples of the same geometry.

$Y_{bone} \approx$ 1010 N/m_2 ρ_{bone} = 1850 kg/m_3
Y_{Al} = 7.0 x 1010 N/m_2 ρ_{Al} = 2700 kg/m_3

Since the geometry of the samples is the same, geometrical factors involved in the inertia parameter in Equation 15.19 will cancel out when we take a ratio. Thus we can write the following equation:

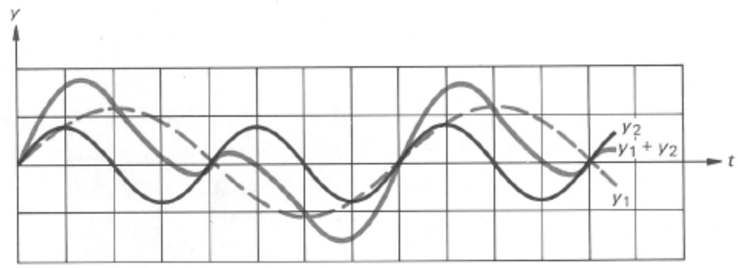
f_{Al}/f_{bone} = SQR RT [(Y_{bone}/ρ_{bone})/ (Y_{Al}/ρ_{Al})]= SQR RT [(1010 x 2700) / (1850 x 7.0 x 1010) ≈ 0.45

When a solid system is subjected to an external periodic force, the Hooke's law model predicts *resonance* just as outlined in Chapter 2. This *resonance* occurs when the frequency of the external force equals a natural frequency of the system. Damping will tend to dissipate the energy into frictional heat and thus decrease the energy that is transferred to the vibration of the solid in resonance. Damping can be used effectively to prevent the destruction of mechanical systems by resonance oscillations.

15.6 Superposition of Simple Harmonic Motions

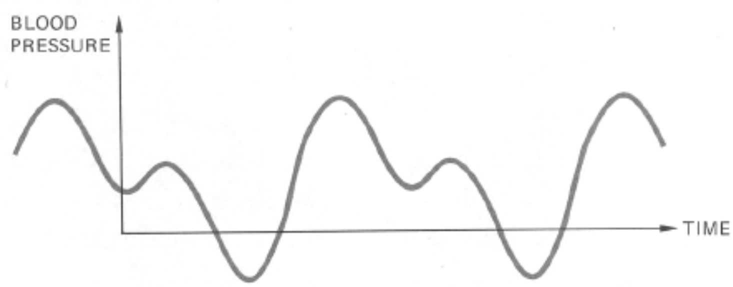
Any periodic motion, regardless of its complexity, can be reduced to the sum of a number of simple harmonic motions by the application of the superposition principle as discussed in Chapter 2. The resultant displacement of a particle at any time t is the vector sum of the separate displacements of the various natural frequency motions (Figure 15.9).

FIGURE 15.9
Superposition of two simple harmonic motions in the same direction.



These component motions are the normal modes of the system, and this analysis of complex vibrations in terms of normal modes is an example of Fourier analysis. For example, a person's blood pressure is periodic and may have a tracing like that shown in Figure 15.10 . The curve is equivalent to the sum of a number of simple harmonic motions. Using modern computer assisted techniques, we can analyze this curve into component parts. A change in the relative contributions can be used in the diagnosis of various heart conditions. Other examples of periodic phenomena that may be used in medical diagnoses are electrocardiograms, respiration graphs, gastric motility tracings, and ballistocardiograms.

FIGURE 15.10
Blood pressure as function of time.



EXAMPLES

1. Consider a particle that is subject to two simple harmonic motions given by $y_1 = 6 \sin \pi t$ and $y_2 = 4 \sin 2\pi t$. Plot $y = y_1 + y_2$. What is the displacement of the particle? See Figure 15.9.

2. Find the resultant motion if a body is subjected to the following SHM: $x = 2 \sin 2\pi t$, $y = 3 \sin \pi t$.

Figure 15.11 shows the resulting motion. This is an example of the patterns called Lissajou's figures that are produced by the superposition of two simple harmonic motions at right angles. These patterns are periodic, and the period is an integral multiple of the periods of the two basic simple harmonic motions.

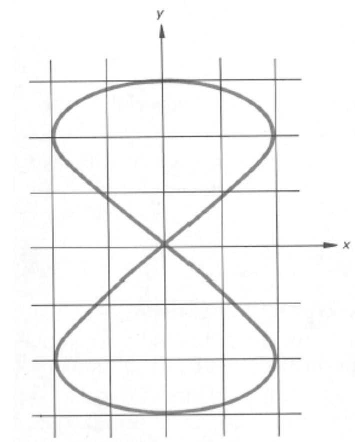


FIGURE 15.11
Lissajou's figure formed by the superposition of two simple harmonic motions at right angles to each other.

ENRICHMENT 15.7 Taylor's Theorem

For many systems that have an equilibrium location it is possible to find the potential energy as some function of the displacement of the system from equilibrium.

$$PE = V(x) \qquad (15.20)$$

where $V(x)$ is some potential energy function that can be written in terms of the displacement x from equilibrium. From the definition of work and its relationship to force, we can show that the force which gives the potential energy function $V(x)$ is

$$F(x) = -dV(x)/dx) \qquad (15.21)$$

Taylor's theorem asserts that if $V(x)$ is a continuous and differentiable function, we can write a series of terms that will equal the function if we take enough terms. The form of the Taylor series is shown below,

$$V(x) = a_o + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ... + a_nx^n + ... \qquad (15.22)$$

where the a_n's are constants. We can use this Taylor series to find the form of the force for our system by differentiating the series

$$F(x) = -V'(x) = -a_1 - 2a_2x = 3a_3x^2 - 4a_4x^3 - ... - na_nx^{n-1} + ...$$

But at equilibrium, all the forces must add to zero, and since $x = 0$ is our equilibrium location: $F(0) = 0 = -a_1$, so $a_1 = 0$.

$$F(x) = -2a_2x - 3a_3x^2 - 4a_4x^3 - ... - na_nx^{n-1} \qquad (15.23)$$

If we choose x much smaller than 1, then x^2 is much, much smaller than x; and x^3 is much, much, much smaller than x^2, and so forth. Therefore, we can find a small value of x, close to x = zero, the equilibrium location, where the first term of the series for $F(x)$ is much larger than all the other terms,

$$F(x) \approx -kx$$

a Hooke's law force. Clearly for this to be correct the constant a_2 must be a positive number, a situation that exists for all potential functions near *stable* equilibrium. Think of the scope of this result: all objects in stable equilibrium will perform simple harmonic motion if displaced only slightly from their equilibrium positions. This is a result that applies as well to planets performing small oscillations about their stable orbits around the sun as it does to molecules vibrating about their equilibrium sites in a crystal.

15.8 Simple Pendulum

We want to examine the motion of a simple pendulum, a point mass m suspended by a weightless, frictionless thread. Under what conditions does a simple pendulum exhibit simple harmonic motion? To demonstrate that a system shows SHM, we need to show only that the force that tends to restore the system to equilibrium is of Hooke's law form. Consider the simple pendulum shown in Figure 15.12. For a displacement of θ from equilibrium position, we consider the forces acting on the mass-the weight of the mass and the tension in the string. The tension in the string is always perpendicular to the direction of motion, and the weight of the ball is always vertical. The component of the weight parallel to the direction of motion is the restoring force, and this component is given by $mg \sin\theta$. The restoring force is always directed toward the equilibrium position. However, it is not of the Hooke's law form but rather

$$F = -mg \sin\theta \qquad (15.24)$$

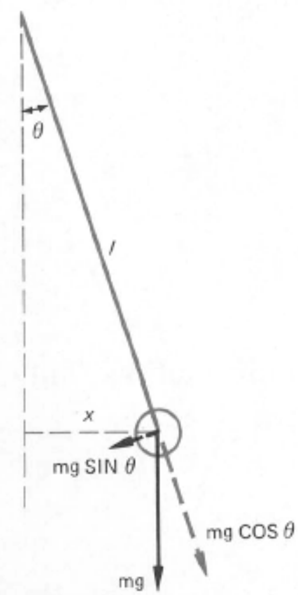


FIGURE 15.12
A simple pendulum. An analysis of the forces acting is shown. Note that the restoring force is proportional to the displacement for small angles.

Let us recall our result from the previous section. It is only close to equilibrium that we can be sure of finding the Hooke's law force, as the following table of displacements θ shows.

θ(°)	θ (rad)	sin θ
0.000	0.0000	0.0000
3.000	0.05236	0.05233
6.000	0.1045	0.1047
9.000	0.1571	0.1564
12.000	0.2094	0.2079
15.000	0.2618	0.2588
18.000	0.3142	0.3090
21.000	0.3665	0.3584

Notice that the angle in radians is almost exactly equal to the sine of the angle for small angles; $\sin\theta \approx \theta$ (rad) for $\theta \ll 1$. Then the restoring force for a small angle becomes $F = -mg\theta$. The value of θ is given by x/l, and the restoring force then becomes $F = mgx/l$. This is of the form $F = -kx$, where $k = mg/l$, which is the correct force relationship for simple harmonic motion. Earlier we found that $k/m = \omega^2$, which reduces to $g/l = \omega^2 = (2\pi f)^2 = (2\pi/T)^2$. Thus, the period for a simple pendulum is given

$T = 2\pi$ SQR RT $[\, l/g \,]$ (15.25)

In general, if a distortion of an elastic body is produced, and the restoring force or torque is proportional to the magnitude of the change, the system will execute simple harmonic motion. If a shaft is twisted through an angle θ, the restoring torque is of the form $t = -k'\theta$ where k' is a constant depending upon the shear modulus and geometry of the shaft. From rotational dynamics we learned that $t = I\alpha$. Equating these values we get

$I\alpha = -k'\theta$ and $\alpha = -k'/I\theta$

and

$\omega^2 = k'/I$ (15.26)

This is the basic relationship for a torsion pendulum and the equation of motion is

$\theta = \theta_o \cos(\omega t + 0)$

where

$\omega =$ SQR RT $[k'/I]$ (15.27)

15.9 Calculus Derivations of SHM Relationships

If the displacement of a particle is represented by the equation $x = A\cos(\omega t + \theta)$, one can find the expression for the instantaneous velocity by finding the derivative relative to t. Thus,

$$v_x = dx/dt = -\omega A \sin(\omega t + \phi)$$

and

$$a_x = dv_x/dt^2 = d^2x/dt^2 = -\omega^2 A \cos(\omega t + \phi)$$

These are the kinematic equations for simple harmonic motion. The general force equation for simple harmonic motion is

$$m\, d^2x/dt^2 = -kx \quad \text{or} \quad d^2x/dt^2 + (k/m)x = 0$$

If we let $\omega^2 = k/m$, this reduces to

$$d^2x/dt^2 + \omega^2 x = 0$$

Any system that has a force equation equivalent to this form will execute simple harmonic motion.

SUMMARY

Use these questions to evaluate how well you have achieved the goals of this chapter. The answers to these questions are given at the end of this summary with the number of the section where you can find the related content material.

Definitions

Circle the correct answer(s) for each question.

1. Simple harmonic motion has the following characteristic(s):

   a. period
   b. linear restoring force
   c. natural frequency
   d. zero amplitude
   e. none of these

2. The period of a simple pendulum is equal to:

   a. time for one swing
   b. amplitude/velocity
   c. 1/frequency
   d. 2π SQR RT [m/k]
   e. none of these

3. The phase angle of a SHM system where $x = A \cos(\omega t + \phi)$ at $t = 0$ is 90°. The correct form of this SHM as a function of time, is:

   a. $A \cos(\omega t + 90^{o})$
   b. $A \cos(\omega t + \pi/2)$
   c. $A \cos(\omega t - 90^{o})$
   d. $A \cos(\omega t - \pi/2)$
   e. $A \cos 90^{o}$

4. Damping of periodic motion always results in:

   a. decreasing amplitude
   b. energy gain in vibration
   c. resonance
   d. energy loss to friction
   e. none of these

UCM and SHM

5. If the frequency of a 0.75-m simple pendulum is 1.5 Hz, the angular frequency on a corresponding reference circle is (in rad/s):

   a. 1.5π
   b. 0.33π
   c. 3π
   d. 0.5π
   e. 2π

6. The expression for the radial acceleration on the reference circle can be expressed in terms of amplitude *A*, period *T*, and maximum speed *v* max of the corresponding SHM as:

   a. $v_{max}A/T$
   b. v_{max}^2/A
   c. A/T^2
   d. v_{max}/T
   e. none of these

7. If you know the force constant of a spring is 100 N/m, the mass on the spring is 1 kg, and the amplitude is 0.04 m, then the period in seconds of the SHM is:

   a. 20π
   b. π/5
   c. 5π
   d. 5/π
   e. π/20

8. For the system described in question 7, the maximum velocity (in m/s) of SHM is:

   a. 100
   b. 0.04
   c. 0.4
   d. 4.0
   e. 0.004

9. For this same system (question 7) the maximum acceleration (m/s^2) of SHM is:

   a. 4
   b. 0.4
   c. 0.04
   d. 100
   e. 25

10. The minimum speed (m/s) of SHM for the system of the question 7 is

    a. 0.04
    b. 0.4
    c. 0.004
    d. 100
    e. 0

11. The maximum displacement for this sytem is:

    a. 0.4 m
    b. 0.04 m
    c. 0.004 m
    d. 100 m
    e. 9.8 m

12. The equation for displacement for this system could be written as

    a. $y = 0.04 \cos(10\pi t)$
    b. $y = 0.04 \cos(10t + \phi)$
    c. $y = 0.04 \sin(10t + \phi)$
    d. $y = 0.04 \sin(10\pi t)$

13. For the spring system in question 7 the maximum PE is

    a. 0.08 J
    b. 0.4 J
    c. 0.16 J
    d. 1.6 J
    e. 0.8 J

14. The kinetic energy is equal to the potential energy when the displacement is

    a. 0.04 m
    b. 0.04 x SQR RT [2] m
    c. 0.04/SQR RT [2] m
    d. SQR RT [2] m
    e. 0

Superposition Principle

15. If two vibrations of the same frequency are superimposed on a system with equal amplitudes, the maximum resultant amplitude could be

    a. zero
    b. SQR RT [2A]
    c. *A*/SQR RT [2]
    d. 2*A*
    e. *A*

16. For the case given in the previous question, the minimum amplitude could be

    a. zero
    b. SQR RT [2] *A*
    c. *A*/SQR RT [2]
    d. 2*A*
    e. *A*

Natural Frequencies

17. The natural frequency of a system is determined in analogy with a spring to be proportional to

    a. SQR RT [inertia parameter/force constant]
    b. SQR RT [force constant/inertia parameter]
    c. amplitude/period
    d. none of these

18. The natural frequency of rod A is 1/2 times the natural frequency of rod B for longitudinal vibrations. If the rods have the same geometry and $\rho_A = 2\rho_B$, then the ratio of Young's moduli for A to B is:

    a. SQR RT [2]
    b. 2
    c. 4
    d. 1/SQR RT [2]
    e. 1/2

Answers

1. a, b, c (Section 15.2)
2. c, d (Section 15.2)
3. b (Section 15.2)
4. a, d (Section 15.5)
5. c (Section 15.2)
6. b (Sections 15.2, 15.3)
7. b (Section 15.3)
8. c (Section 15.3)
9. a (Section 15.3)
10. e (Section 15.3)
11. b (Section 15.3)
12. b, c (Section 15.3)
13. a (Section 15.3)
14. c (Section 15.3)
15. d (Section 15.6)
16. a (Section 15.6)
17. b (Section 15.5)
18. c (Section 15.5)

ALGORITHMIC PROBLEMS

Listed below are the important equations from this chapter. The problems following the equations will help you learn to translate words into equations and to solve single-concept problems.

Equations

$F = -kx = -m\omega^2 x$ (13.3)

$x = A\cos\theta$ (15.1)

$x = A\cos(\omega t + \phi)$ (15.2)

$a_x = -\omega^2 x$ (15.6)

$v_x = -\omega A\sin(\omega t + \phi)$ (15.7)

$a_x = -\omega^2 A\cos(\omega t + \phi)$ (15.7)

$\omega = 2\pi f = 2\pi, f = n = \omega/2\pi = 1/(2\pi)$ *SQR RT*(k/m) (15.8)

$T = 1/f = 2\pi/\omega = 2\pi$ *SQR RT*(m/k) (15.9)

$T = 2\pi$ *SQR RT*$(-x/a)$ (15.11)

$f = (1/2\pi)$ *SQR RT*$(-a/x)$ (15.11)

$PE = (½)kx^2$ (15.12)

$E = (½)kA^2 = (½)mv^2 + (½)kx^2$ (15.13, 15.14)

$v^2 = (k/m)(A^2 - x^2) = \omega^2(A^2 - x^2)$ (15.15)

$v_{max}^2 = (k/m)A^2 = \omega^2 A^2$ (15.16)

$T = 2\pi$ SQR RT(l/g) (15.25)

Problems

1. The force constant of a spring is 10 N/m. Find the period of a 100-g mass on the end of this spring.
2. Find the maximum velocity of the mass in problem 1 if the amplitude of oscillation is 2.0 cm.
3. Find the velocity of the mass in problem 2 when it is 1 cm from its equilibrium position.
4. A mass on the end of a spring is released from a point 2 cm from its equilibrium position. The frequency of oscillation is 4 Hz. Write the equation for the position of the mass as a function of time.
5. The period of a simple pendulum is 2.00 sec. Find the length of the pendulum.
6. Find the maximum energy stored in the spring of problem 1 when it is compressed 2 cm from its equilibrium position.

Answers

1. 0.63 s
2. 20 cm/s
3. 17 cm/s
4. $0.02 \cos (8\pi t)$ m
5. 0.993 m
6. .02 J

EXERCISES

These exercises are designed to help you apply the ideas of a section to physical situations. When appropriate the numerical answer is given in brackets at the end of the exercise.

Section 15.2

1. Show in tabular form the sign of displacement, velocity, and acceleration for each quadrant, i.e., 0
2. A body is executing simple harmonic motion, and the displacement is given by $x = 5 \cos 3\pi t$. Plot the displacement, velocity, and acceleration for two complete periods.
3. If simple harmonic motion has an amplitude of 10 cm, what is the maximum and minimum change in position in one-fourth of the period? [14 cm, 0 cm]
4. A body is vibrating with simple harmonic motion of amplitude 15 cm and a frequency of 2.0 Hz.

   a. What is its maximum acceleration and maximum velocity?
   b. What is its acceleration and velocity for a displacement of 12 cm?
   c. How long does it take to go from equilibrium position to a displacement of 9.0 cm?

   [a. $240\pi^2$ cm/s^2, 60π cm/sec; b. - $192\pi^2$ cm/s^2, ± 36π cm/s; c. 0.051 s]

5. A particle is moving along the χ-axis in accordance with the following:
   $x = 5 \cos (4\pi t + \pi/3)$ cm

   What is the

   a. amplitude of motion?
   b. period of motion?
   c. frequency of motion?
   d. time for $\chi = 0$?
   e. time for velocity to be 0?
   f. time for acceleration to be a maximum?

   [a. 5 cm; b. 1/2 s; c. 2 Hz; d. 1/24 s, 7/24 s, 13/24 s; e. 1/6 s, 5/12 s; f. same as e]

Section 15.3

6. Assume the piston in an automobile engine is executing simple harmonic motion. The length of the stroke (double the amplitude) is 10 cm, and the engine is running at 300 rpm.

   a. What is the acceleration at the end of the stroke?
   b. If the piston has a mass of 0.50 kg, what is the maximum restoring force acting on the piston?
   c. What is the maximum velocity of the piston?
   d. What is the position of the piston as a function of time if it is at the top of the stroke at $t = 0$?

   [a. $500\pi^2$ cm/s^2; b. $250\pi^2$ N; c. 50π cm/s; d. $x = 5 \cos 10\pi t$ cm]

Section 15.4

7. A 2.0-kg mass is attached to a spring with a force constant of 98 N/m. The mass is resting on a frictionless horizontal plane as a horizontal force of 9.8 N is applied to the mass, and it is then released.

   a. What is the amplitude of SHM?
   b. What is its period?
   c. What is the total energy of the SHM?
   d. What is the maximum velocity of the vibrating mass?
   e. What is the PE and KE of the mass when it is 5 cm from equilibrium position?

   [a. 0.10 m; b. $2\pi/7$ s; c. 0.49 J; d. 0.7 m/s; e. PE = 0.12 J, KE = 0.37 J]

Section 15.5

8. Compare the natural frequencies of longitudinal vibrations for rods of the same geometry made of steel and aluminum. Use values from Table 8.1 and13.1.

   [f steel/ fAl = .99]

Section 15.6

9. A particle is subjected simultaneously to two simple harmonic motions of the same frequency and direction in accordance with the following equations:
   $y_1 = 6 \sin \omega t$ cm $y_2 = 8 \sin (\omega t + \pi/3)$ cm $\omega = 2\pi$
   Find the amplitude of the resultant motion, and show it in graphical form. [12.17 cm]

10. A particle is subjected simultaneously to two simple harmonic motions in the same direction in accordance with the following equations:
    $y_1 = 8 \sin 2\pi t$ $y_2 = 4 \sin 6\pi t$
    Show graphically the resultant path of the particle.

11. A particle is simultaneously subjected to two simple harmonic displacements at right angles to each other. Show the pattern of the particle in the xy plane. The displacements are: $x = 5 \cos \pi t$ $y = 3 \sin (3\pi t)$

PROBLEMS

Each problem may involve more than one physical concept. A problem requiring material from the enrichment section is marked with a dagger †. The answer is given in brackets at the end of the problem.

12. A frame with seats, which weighs 15,680 N, is mounted on springs, and it is executing vertical simple harmonic motion with a frequency of 4 Hz.

    a. What is the force constant of this system of springs?
    b. If four students of total mass 260 kg are seated in the frame,
    what is the frequency of oscillation? [a. 1.01 x 10^6 N/m; b. 3.71 Hz]

†13. In a laboratory experiment a student observes that two simple pendulums of different lengths have periods that differ by 10 percent. The student then asks, "What is the percentage difference in length?" What is the answer? Can you make a general statement? [21 percent]

†14. A pendulum clock should have a period of 2 s to keep accurate time. Observation shows that the clock loses 10 min per day. Assume that the pendulum system behaves as a simple pendulum. What changes should be made and how much? [decrease length of pendulum by 1.38 cm]

†15. The acceleration due to gravity on the moon is about one-fifth that on the earth. What is the period of a simple pendulum on the moon, if it has a period of 2 s on the earth? [4.5 s]

†16. A spring driven clock has a period of 2 s on earth. What is the period on the moon? [2 s]

17. A simple pendulum can be used to determine experimentally the acceleration due to gravity. What is the acceleration due to gravity at the place where a 1-m simple pendulum has a period of 2 s? [$g = \pi^2$ m/s^2]

18. An arrow of length 0.6 m is rotated about a vertical axis through its tail with an angular velocity of π rad/s. The motion of the tip of the arrow is to be projected upon a diameter of the circle. Find the following:

a. the amplitude of its motion
b. its period
c. the maximum acceleration
d. the maximum velocity
e. the acceleration and velocity when the arrow tip is 0.3 m from the center of its projected path. [a. 0.6 m; b. 2 s; c. 5.92 m/s^2; d. 0.6π m/s; e. 2.96 m/s^2, 1.63 m/s]

†19. A child's swing has a period of 5.0 sec and an amplitude of 1 m.

a. What is the angular speed of an imaginary particle moving in a reference circle to represent this vibration?
b. What is the maximum speed of the swing seat?
c. What is the length of rope from point of support to the seat? [a. 0.4π rad/s; b. 4π/10 m/s; c. 6.21 m]

20. Assume a person's heart is executing SHM with 66 beats per minute. If its amplitude is 3 mm, what is its maximum velocity and acceleration? [2.1 cm/s, 14.3 cm/s^2]

†21. A pendulum bob consists of a 1.00-kg ball hung on a string 3m long. If it is drawn back 20 cm from equilibrium position and released, what is its period of motion? What will be its kinetic energy as it passes through the middle of its swing? What are its PE and KE for a displacement of 10 cm? [3.5 s, 0.685 J, PE = 0.196 J, KE = 0.489 J]

†22. Suppose a force $F = F_0 \cos\omega t$ is applied to a spring-mass system with a natural frequency ω_0. The equation of motion for this system will be

$$F_o \cos \omega t - kx = m d^2x/dt^2$$

Show that a solution for this equation is:

$$x = F \cos \omega t \;/\; \{m(\omega_o^2 - \omega^2)\}$$

Note that as $\omega \rightarrow \omega_0$, the amplitude becomes very large. This is characteristic of undamped resonant oscillations.